Competitive Advertising and Pricing*

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Abstract

We consider an oligopoly model in which each firm chooses not only its price but also its advertising strategy regarding how much product information to provide. To highlight firms' strategic incentives, we impose no structural restriction on feasible advertising content, so that each firm can disclose or conceal any information. We provide a general characterization of the equilibrium advertising content and show that intense competition induces firms to provide precise product information. We also demonstrate that strategic advertising has ambiguous implications for market prices and consumer surplus.

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1 Introduction

One of the central questions in the economics of advertising is how much, and what, product information a firm would provide for consumers.[1] Offering more product information can allow the firm to be more aggressive in pricing or try some form of price discrimination. However, it comes at the cost of losing some consumers who do not find the revealed product characteristics

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[1]We restrict attention to this unbiased information transmission role of advertising. However, the literature has identified several other roles. See Bagwell (2007) and Renault (2015) for comprehensive overviews of the literature.

appealing. This fundamental trade-off has been extensively studied in the monopoly context.[2] A common insight in the literature is that in the absence of varying advertising costs, a monopolist wishes to provide either no information or full information: the former enables her to serve all consumers without charging too low a price, while the latter allows her to extract the most from high-value consumers.

In this paper, we study firms' incentives to provide product information in an oligopoly environment. We employ the canonical random-utility discrete-choice framework of Perloff and Salop (1985): there are $n(\geq 2)$ firms that engage in Bertrand competition for a consumer whose value for each product is independently and identically drawn from a certain distribution F. We extend the basic framework to allow for strategic advertising by the firms: each firm chooses how much information to provide about its own product.[3] Therefore, the firms compete not only on price but also through advertising (information provision).

Unlike most classical studies on advertising, we impose no structural restriction on the set of feasible advertising strategies and allow each firm to disclose or conceal any information.[4] As is well known in the literature on information design (e.g. Ostrovsky and Schwarz, 2010; Gentzkow and Kamenica, 2016; Kolotilin et al., 2017; Roesler and Szentes, 2017), with a risk-neutral consumer, this fully flexible advertising can be modeled as each firm being able to choose any mean-preserving contraction (MPC hereafter) G_i of the true value distribution F.[5] Since it is as if each firm chooses the consumer's value distribution for its product, our model can be interpreted as one in which the consumer's value distribution—exogenously given in Perloff and Salop (1985)—is endogenously determined by firms' advertising choices.

We first demonstrate that intense competition induces—or, forces—firms to provide precise

[2] Lewis and Sappington (1994) is a seminal contribution to this literature. Subsequent important works include Che (1996) (return policies as a means of facilitating consumer experimentation), Ottaviani and Prat (2001) (the value of revealing a signal associated with the buyer's private signal), Anderson and Renault (2006) (optimal advertising for search goods), Johnson and Myatt (2006) (U-shaped profit function based on the rotation order), and Roesler and Szentes (2017) (buyer-optimal signal for experience goods). See also Boleslavsky et al. (2017), who study the interaction between advertising ("demonstration") and pricing in the entry game context (where consumers' values for the incumbent's product are known, while their idiosyncratic values for the entrant's product are unknown).

[3] We do not allow for "comparative advertising," in which a firm not only controls its own product information but can also provide information about rivals' products. See Anderson and Renault (2009) for an important contribution on comparative advertising. We also assume that advertising is costless; this assumption enables us to focus on strategic incentives. It would be an interesting but technically challenging exercise to introduce explicit advertising (information) costs into our model.

[4] See Anderson and Renault (2006) for an earlier application of this general formulation to advertising. The assumption can be interpreted as each firm having access to numerous advertising channels and fine-grained information on various product attributes, which enables them to adjust advertising contents without restriction. However, note that the main advantage of our information design approach is not that its full flexibility is particularly realistic, but that it enables us to focus on firms' strategic incentives, not subject to any structural assumptions.

[5] Intuitively, each firm can induce a degenerate distribution by revealing no product information and F itself by revealing all product information. It obtains any distribution between these two extremes by selectively disclosing information but cannot induce a more dispersed distribution than F.

product information. Specifically, we consider a restricted game, termed *the advertising game*, in which the firms compete by choosing their advertising strategies given symmetric prices and show that the full information equilibrium—in which each firm chooses the underlying distribution F—exists if and only if F^{n-1} is convex over its support. Notice that F^{n-1} represents the distribution of the consumer's best alternative to a given product. So from an individual firm's perspective, F^{n-1} being convex means that the consumer likely has an attractive outside option. In this case, a firm should maximize the probability that the consumer has a sufficiently high value to its product, which can be done by providing precise information in our informative-advertising environment.

One important corollary of the above result—in fact, this paper's main economic insight—is that the full information equilibrium exists whenever there are sufficiently many firms.[6] Mathematically, this is simply because F^{n-1} is convex for n sufficiently large, irrespective of the shape of F. Economically, when there are many competitors, a firm should aggressively take a risk and maximize the probability that the consumer highly values its product. Importantly, this result is robust along multiple dimensions; in Section 6, we show that even if the result may not hold for any finite n, the equilibrium distribution G^* often converges to F as n tends to infinity.

In general, the unique symmetric equilibrium advertising strategy, denoted by G^*, features the following two properties: (i) $(G^*)^{n-1}$ *is convex*, and (ii) *locally, either* G^* *coincides with* F *or* $(G^*)^{n-1}$ *is linear*. The first property ensures that a firm has no incentive to contract its distribution by providing less information: if $(G^*)^{n-1}$ is convex, then a firm is "risk-loving" and so unwilling to reduce risk. The second property prevents the opposite incentive to disperse its distribution by providing more information: a firm would not disclose more information if there is no further information to reveal (i.e., $G^* = F$), or it is "risk neutral" and so has no incentive to do so (i.e., $(G^*)^{n-1}$ is linear). These two necessary properties are effectively sufficient for equilibrium advertising and provide a tractable method to construct or verify the equilibrium distribution.[7]

We then evaluate the economic effects of strategic advertising by comparing the equilibrium outcome to that of the full information benchmark; this comparison is not only of theoretical interest but also has direct implications for policies regarding firms' disclosure requirements. Clearly, strategic advertising reduces social surplus through information loss. However, it has ambiguous implications for both producer and consumer surplus; each can increase or decrease depending on the shape of F. We analyze this ambiguity by establishing a new result (Proposition 3) that reveals a crucial distributional property for the equilibrium price in the Perloff-Salop model.

One technical aspect worth contemplating regards the existence of pure-price equilibria.[8] In

[6] As elaborated shortly, this result complements the seminal result by Ivanov (2013).

[7] For example, the aforementioned full information result is immediate from this general characterization. If F^{n-1} is concave then $(G^*)^{n-1}$ should be linear over its support. See Section 4 for more examples.

[8] The existence of pure-price equilibrium is a classical problem in the Perloff-Salop model. The main difficulty is that each firm's best response depends on the entire distribution, so it may not be well behaved without some strong

our model, a firm can engage in a compound deviation, changing both its price and advertising strategy from its equilibrium pair (p^*, G^*). This suggests that given the same F, a pure-price equilibrium may exist in the Perloff-Salop model, but not in ours. Interestingly, the opposite case also can arise: a pure-price equilibrium exists in our model, but not in the corresponding Perloff-Salop model. This is because a firm's deviation incentive depends on whether the other firms play F or G^*. Finally, we show that, despite these complications, sufficient regularity on F ensures the existence of pure-price equilibrium in our model.

Related literature. As explained above, the interaction between advertising and pricing has been extensively studied in the monopoly context but not in oligopoly settings. To the best of our knowledge, the only exception is Ivanov (2013). He also adopts the Perloff-Salop framework but considers the case in which firms are endowed with a restricted set of advertising strategies that are rotation-ordered in the sense of Johnson and Myatt (2006). Our full information result for n sufficiently large (Corollary 1) can be interpreted as showing that the same economic conclusion can be drawn without restriction on feasible advertising content. Our information design approach permits a more comprehensive characterization than his structural approach: Ivanov (2013) shows only that the maximal information equilibrium exists when n is sufficiently large. By contrast, we provide a necessary and sufficient condition for full information disclosure and also characterize the equilibrium structure when the full information equilibrium does not exist.

Our advertising game can be interpreted as an information disclosure game with multiple senders. There are two strands in that literature, one in which the senders have access to the same information or the full state of nature (e.g., Gentzkow and Kamenica, 2017; Li and Norman, 2021) and another in which each sender controls only his own information. Our model belongs to the latter category because each firm only provides information about its own product.

Our equilibrium characterization of the advertising game closely resembles that of Ostrovsky and Schwarz (2010). They consider a competitive information disclosure problem in the context of school grading policies and also find that the equilibrium aggregate distribution has an alternating structure between coinciding with the primitive aggregate distribution and being locally linear. The main difference is that they adopt a Walrasian equilibrium approach, assuming each school chooses its transcript structure while taking the desirability mapping (the aggregate outcome) as given, and the desirability mapping should be consistent with schools' choices in equilibrium. Their approach enables them to consider a more general environment with heterogeneous schools and any reward structure. However, it is unsuitable for our oligopoly problem, in which an individual firm's limited market power is a key element. For example, their approach cannot be used for comparative static results regarding the number of firms.

regularity on F. See Caplin and Nalebuff (1991) and Quint (2014) for some important contributions.

Other papers in the same category, such as Boleslavsky and Cotton (2015, 2018) and Au and Kawai (2020, 2021), consider a fully strategic setting but assume discrete distributions. They also show that the equilibrium distributions are such that each player faces a locally linear value function.[9] Our analysis reveals that the equilibrium structure—in particular, the relationship between F and the equilibrium distribution G^*—is more transparent when F is a continuous distribution. It also allows us to directly link our work to the large discrete-choice literature, most of which considers continuous distributions.

We benefit from recent technical developments in the literature on information design. In particular, we utilize a tractable verification technique developed by Dworczak and Martini (2019) (DM hereafter).[10] Our analysis differs from DM's in two ways. First, they consider an exogenously given programming problem, while we search for Nash equilibrium in a strategic environment, so each firm's problem is endogenously determined in our model. Second, in our model, an individual firm chooses not only its advertising strategy but also its price, which necessitates combining DM's result with the traditional pricing argument.

More broadly, this paper belongs to the growing literature that combines information design with classical industrial organization problems. Most papers in this literature consider the simpler monopoly setting,[11] but there is growing interest in oligopoly problems.[12] Particularly related to our paper is Armstrong and Zhou (2022), who also consider Bertrand competition under horizontal differentiation. They consider an independent information designer with full access to the consumer's value profile and solve for the *optimal* information structures (maximizing producer surplus or consumer surplus); by contrast, we characterize the *equilibrium* information structure that arises when each firm controls the consumer's information about its own product. Their paper and ours complement each other not only by studying conceptually different problems, but also by employing different frameworks and techniques: they consider Hotelling's linear city model (effectively) and analyze the information design problem given firms' equilibrium pricing incentives. By contrast, we build upon Perloff and Salop's random-utility model and derive the equilibrium

[9] The same linear property of the equilibrium value function appears in the literature on risk-taking contests (see, e.g., Myerson, 1993; Ray and Robson, 2012). This similarity is not surprising because mathematically, the game of competitive information disclosure differs from risk-taking contests only in that the former features the (more demanding) MPC constraint while the latter involves the mean constraint.

[10] Other relevant contributions include Gentzkow and Kamenica (2016), Kolotilin (2018), Dworczak and Kolotilin (2019), Dizdar and Kováč (2020), and Ivanov (2021).

[11] This literature is already too extensive to summarize here. See Anderson and Renault (2006), Bergemann et al. (2015), and Roesler and Szentes (2017) for some seminal contributions.

[12] For instance, Armstrong and Vickers (2019) and Shi and Zhng (2022) study the welfare implications of market segmentation structures with price-discriminating firms, and Condorelli and Szentes (2022) characterize the set of all possible equilibrium outcomes in Cournot competition given maximal willingness to pay and maximal total demand. Elliott et al. (2022) consider a general Bertrand model in which the firms receive information about consumers and characterize the producer-optimal and the consumer-optimal information structures: their work can be interpreted as an oligopoly extension of Bergemann et al. (2015).

information structure based on firms' advertising incentives.

The remainder of this paper is organized as follows. Section 2 introduces the formal model. Section 3 analyzes the full information equilibrium. Section 4 provides a general characterization for the advertising game. Section 5 studies the price and welfare effects of strategic advertising and addresses the equilibrium existence problem of our full game. Section 6 considers three variants of our baseline model and examines the robustness of our full information result. Section 7 concludes. All the proofs not in the main text are relegated to Appendix A unless noted otherwise.

2 The Model

Our model builds upon the random-utility oligopoly framework of Perloff and Salop (1985). There are $n(\geq 2)$ firms and a risk-neutral consumer.[13] Each firm, indexed by i, supplies a product with marginal cost normalized to zero, and the consumer demands one unit. The firms' products are horizontally differentiated. Specifically, the consumer's value for each product is independently and identically drawn according to the distribution function F. We assume that F has compact and convex support $[\underline{v}, \overline{v}]$ with mean μ_F, and its density f is positive and continuously differentiable over $[\underline{v}, \overline{v}]$. We also assume that both f and f' are bounded, and f has a finite number of peaks.

Firm i's strategy consists of its price p_i and advertising strategy. For the latter, we assume that each firm can choose any mean-preserving contraction (MPC, hereafter) G_i of the underlying distribution F; that is, the set of feasible advertising strategies is given by the set of all MPCs of F, henceforth denoted by MPC(F). By now, it is well-known that this MPC formulation captures, without loss, full flexibility in information choice when an agent's decision depends on her posterior mean. It is suitable for our model, because the consumer's purchase decision relies on her expected values for the products and their prices. Note that providing full information corresponds to choosing F itself, and more information increases G_i in convex order.

The timing of the game is as follows. First, the firms simultaneously choose their prices $p_i \in \mathcal{R}_+$ and advertising strategies $G_i \in$ MPC(F).[14] Second, the consumer draws her interim value for each product i according to G_i. Her interim values are independent across the products. Finally, the consumer decides which product to purchase based on prices and her interm values. We assume that the consumer must purchase one of the products, so that she purchases product i if and only if

[13] The model can be interpreted as one with a continuum of ex ante homogeneous consumers. Our single-consumer setup allows us to avoid a measurability issue inherent in the continuum model (see, e.g., Sun, 2006).

[14] One may consider the sequential cases in which the firms choose prices and then advertising strategies, or vice versa. While the most appropriate timing assumption depends on the context, our simultaneity assumption reflects an agnostic view about the timing. Also, our model is relevant to the sequential cases if the firms' first actions remain private. Finally, our assumption simplifies the technical analysis, as it does not involve numerous subgames (following each price or advertising strategy profile) in the sequential case.

$v_i - p_i > v_j - p_j$ for all $j \neq i$.[15]

We study symmetric pure-price equilibria of this market game. Since the consumer's purchase problem is straightforward, we focus on Nash equilibria played by the firms. Let $D(p_i, G_i, p, G)$ denote firm i's demand—the probability that the consumer purchases product i—given its strategy (p_i, G_i) and the other firms' common strategy (p, G). Given the consumer's optimal choice rule,[16]

$$D(p_i, G_i, p, G) = Pr\{v_i - p_i > v_j - p, \forall j \neq i\} = \int G(v_i - p_i + p)^{n-1} dG_i(v_i). \tag{1}$$

A tuple (p, G) is a *symmetric pure-price equilibrium* if (p, G) solves the following individual firm's profit maximization problem:

$$\max_{(p_i, G_i)} \pi(p_i, G_i, p, G) \equiv p_i D(p_i, G_i, p, G) \text{ s.t. } G_i \in \text{MPC}(F).$$

We refer to a symmetric pure-price equilibrium as an *equilibrium* unless it creates confusion.

It is useful to define a game in which the firms choose only their advertising strategy G_i, taking the symmetric price as given. We refer to this restricted game as the *advertising game*. By definition, if (p, G) is an equilibrium in our full game, then G must be an equilibrium in the advertising game. In other words, G being an equilibrium in the advertising game is a necessary condition for it to constitute an equilibrium in the full game. Note that in the advertising game, an individual firm's demand in (1) simplifies to

$$D^A(G_i, G) \equiv D(p, G_i, p, G) = \int G(v_i)^{n-1} dG_i(v_i), \tag{2}$$

and each firm can be interpreted as to maximize its demand $D^A(G_i, G)$.

3 Full information Equilibrium

We first study the case in which the firms *choose* to provide full product information. Specifically, we characterize a condition under which F itself is an equilibrium of the advertising game.

In the advertising game, the full information equilibrium exists if and only if

$$D^A(F, F) = \int F(v)^{n-1} dF(v) \geq D^A(G_i, F) = \int F(v)^{n-1} dG_i(v) \text{ for any } G_i \in \text{MPC}(F).$$

[15] See Section 6.1 for a detailed discussion on the case where the consumer can choose not to purchase any product. As shown below, in any equilibrium of this paper, the probability that the consumer is indifferent among multiple products is negligible. Therefore, for notational and expositional simplicity, we ignore the event throughout the paper.

[16] Note that our derivation of $D(p_i, G_i, p, G)$ does not account for the possibility of atoms in G. This significantly simplifies the notation but incurs no loss of generality because the equilibrium distribution has no atom.

One evident case in which this condition holds is when the integrand F^{n-1} is convex: in this case, the firm is "risk-loving" and so prefers F to any MPC of it (see Proposition 6.D.2 in Mas-Colell et al., 1995). In fact, it is the only relevant case, as formally stated in the following result.

Proposition 1 *In the advertising game, the full information equilibrium, in which all firms choose F, exists if and only if F^{n-1} is convex over its support $[\underline{v}, \overline{v}]$.*

To see why convexity of F^{n-1} is necessary for the full information equilibrium, consider the polar opposite case in which F^{n-1} is strictly concave. In that case, provided that the other firms choose F, an individual firm is "risk-averse" and so prefers any $G_i \in \text{MPC}(F)$ to F itself. In fact, it is easy to see that the degenerate distribution δ_{μ_F} is the firm's optimal choice. The desired result follows from the fact that this logic applies whenever F^{n-1} has a non-convex region, as an MPC that concentrates probability over the non-convex region would outperform F.

For an economic intuition, notice that F^{n-1} represents the distribution of the consumer's best alternative to product i. Convex F^{n-1} means that its density is increasing in v, so the consumer likely has an attractive outside option. In this case, the firm should aggressively take a risk and maximize the probability of offering the highest possible value to the consumer; despite its risk of potentially giving a low value to the consumer, it is more profitable, on average, than taking no risk and offering an mediocre value for sure. When F^{n-1} is convex, this argument continues to apply from $\overline{v}$ to $\underline{v}$, resulting in the optimality of full information provision.

If F^{n-1} is not convex and so does not have increasing density, then the above "unraveling" argument does not survive the entire support $[\underline{v}, \overline{v}]$, in which case F is no longer a best response to F^{n-1}. Note that when F is not convex, characterizing a firm's best response to F^{n-1} is not sufficient for *equilibrium* characterization, because a firm's not playing F changes the other firms' problems and in equilibrium each firm should best respond to the other firms' equilibrium strategies. This equilibrium characterization is the main subject of Section 4.

Although simple to establish, Proposition 1 has an important and robust implication on the effects of competition on strategic advertising, as formally reported in the following result.

Corollary 1 *If n is sufficiently large, then the full information equilibrium necessarily exists in the advertising game.*

Proof. We show that F^{n-1} is convex for n sufficiently large. Let $\varepsilon \equiv \min_{v\in[\underline{v},\overline{v}]} f(v)$ and $M \equiv \max_{v\in[\underline{v},\overline{v}]} |f'(v)|$. Since f is positive and f' is bounded on $[\underline{v}, \overline{v}]$, $\varepsilon > 0$ and $M < \infty$. Then,

$$\left(F(v)^{n-1}\right)'' = (n-1)F(v)^{n-3}\left((n-2)f(v)^2 + F(v)f'(v)\right) \geq (n-1)F(v)^{n-3}((n-2)\varepsilon^2 - M).$$

The result follows because the last expression is always positive for n sufficiently large. ■

Technically, this result holds because the power function x^{n-1} becomes more convex as n rises, so if n is sufficiently large then F^{n-1} is convex, irrespective of the detailed shape of F. Economically, this is precisely due to the effects of competition on firms' advertising incentives. When there are many competitors, the probability of each firm's winning the consumer is low. In that case, an individual firm should be aggressive and attempt to give the highest value to the consumer; otherwise, the consumer would almost surely choose another product.

Two remarks are in order. First, Corollary 1 is robust along multiple dimensions. In Section 6 and Appendix G, we consider various extensions of the model and show that the equilibrium distribution—although it may not coincide with F for any finite n—converges to F as n tends to infinity. Second, imposing certain regularity on F, a result stronger than Corollary 1 holds, namely, that firms provide more information as n increases (see Corollary 2).

4 Competitive Advertising: General Characterization

This section provides a general characterization for competitive advertising. In particular, we determine the general structure of the equilibrium advertising strategy, which we denote by G^*.

Clearly, G^* should be an individual firm's optimal advertising strategy when all other firms choose G^*. This means that G^* is a fixed point that solves

$$\max_{G_i \in \text{MPC}(F)} D^A(G_i, G^*) = \int_{\underline{v}}^{\overline{v}} G^*(v)^{n-1} dG_i(v).$$

Exploiting this fixed-point nature of G^*, we uncover two necessary properties of G^*.

Lemma 1 *$(G^*)^{n-1}$ is convex over its convex support.*

If $(G^*)^{n-1}$ is not convex anywhere, then it is optimal for a firm to concentrate all local probability mass on one point, just as the degenerate distribution is optimal when $(G^*)^{n-1}$ is concave. The resulting optimal solution has a mass point, and thus cannot coincide with the given G^*. Intuitively, in equilibrium no firm should have an incentive to contract its distribution by providing less information. Convexity of $(G^*)^{n-1}$ ensures that by rendering an individual firm "risk-loving."

Lemma 1 implies that locally, $(G^*)^{n-1}$ is either strictly convex or linear. Consider any interval over which $(G^*)^{n-1}$ is strictly convex. For the same reason as given in Section 3, a firm's (local) best response is to disperse its distribution as much as possible by providing full information. This suggests that G^* must coincide with F over the interval. Now consider an interval over which $(G^*)^{n-1}$ is linear. Over the interval, an individual firm is "risk-neutral" and so indifferent over all distributions with the same mean. Therefore, the given G^*—constructed so that $(G^*)^{n-1}$ is linear over the interval—is an individual firm's local best response provided that it is an MPC of F. This yields the following second necessary property of G^*.

Lemma 2 *For almost all $v \in supp(G^*)$, there exists $\varepsilon > 0$ such that either G^* coincides with F or $(G^*)^{n-1}$ is linear over $(v-\varepsilon, v+\varepsilon)$. Over each interval on which $(G^*)^{n-1}$ is linear, G^* must be an MPC of F.*[17]

Intuitively, in equilibrium a firm should have no incentive to disperse its distribution by providing more information, or should not be able to do so even if it wants. Given Lemma 1, the former holds only when $(G^*)^{n-1}$ is linear, so an individual firm is "risk-neutral." The latter corresponds to the case where $G^* = F$; in this case, the MPC constraint binds, so a firm simply cannot disperse its distribution further.

The following theorem states that the necessary conditions in Lemmas 1 and 2 are in fact sufficient for equilibrium advertising, and the advertising game always has a unique equilibrium.

Theorem 1 *A distribution $G^* \in MPC(F)$ is a symmetric equilibrium of the advertising game if and only if it satisfies Lemmas 1 and 2. For each F, such G^* uniquely exists.*

Given the previous discussions, it is natural that Lemmas 1 and 2 are sufficient for the equilibrium advertising strategy: Lemma 1 implies that no firm has an incentive to contract its distribution, while Lemma 2 ensures that no firm benefits from dispersing its distribution. The existence and uniqueness of G^* follow from our constructive proof in Appendix C; we show that starting from $\underline{v}$ and proceeding forward, there is a unique way to construct an MPC of F that satisfies Lemmas 1 and 2. For the essence of our construction, see the proof of Corollary 2 below.

Theorem 1 immediately strengthens our results in Section 3: if F^{n-1} is convex, then the full information equilibrium is the *unique* symmetric equilibrium of the advertising game. In addition, firms *necessarily* provide full product information when n is sufficiently large.

The following result applies Theorem 1 to the canonical case where F^{n-1} has strictly quasi-concave density and shows that in that case, the convergence of G^* to F is particularly transparent.

Corollary 2 *(1) If F^{n-1} has strictly quasi-concave density, then there exists $v^* \in [\underline{v}, \overline{v}]$ such that $G^*(v) = F(v)$ for $v \leq v^*$ and $(G^*)^{n-1}$ is linear above v^*.*
(2) Suppose F^{n-1} has strictly quasi-concave density for any n.[18] *Then as n increases, v^* rises and G^* increases in convex order.*

[17] If G^* has a strictly smaller support than F, then the MPC condition over the last interval with linear $(G^*)^{n-1}$ should hold over the interval including $\overline{v}$, not just over the linear interval. For example, in Figure 1, G^* is an MPC of F over $[v^*, \overline{v}]$, not over $[v^*, \overline{v}^*]$.

[18] This assumption is satisfied by many standard distributions. In particular, it holds whenever f is log-concave.

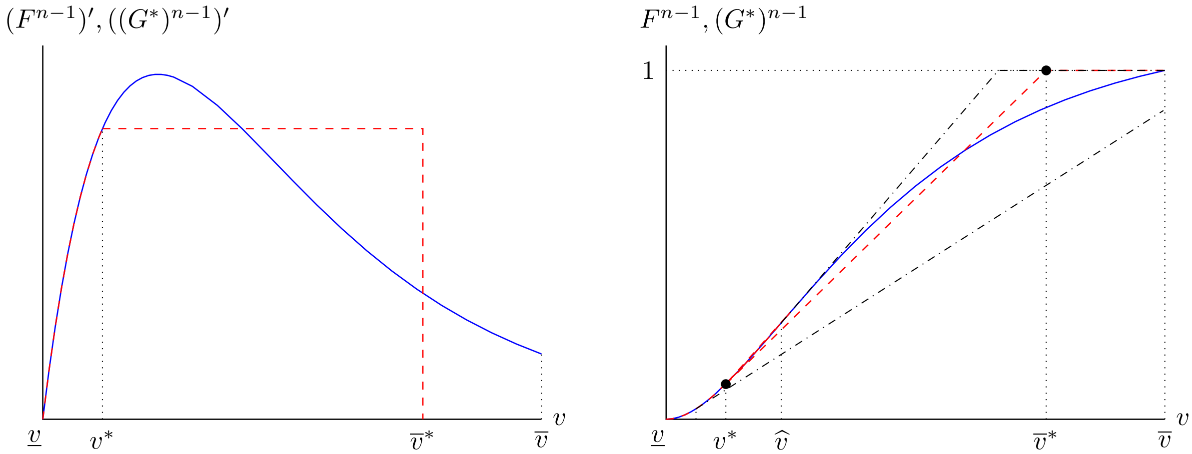

Figure 1: Equilibrium advertising strategy when F^{n-1} has quasiconcave density. In both panels, the blue solid curves represent F, while the red dashed curves depict G^*. In the right panel, the dashdotted curves show how the linear region of $(G^*)^{n-1}$ depends on its starting point, v^*. In this figure, $n = 3$ and $F(v) = (1 - e^{-v})/(1 - e^{-\overline{v}})$ over $[0, 3]$

Proof. We prove (1) here and relegate the proof of (2) to Appendix A. For (1), it suffices to show that there exists $G^* \in \text{MPC}(F)$ with the stated cutoff structure. We can set $v^* = \underline{v}$ if G^* such that $(G^*)^{n-1}$ is linear over its support belongs to $\text{MPC}(F)$.[19] Otherwise, let $\widehat{v}$ denote the unique peak of $(F^{n-1})'$, and continuously raise v^* from $\underline{v}$ to $\widehat{v}$. This lowers G^* in the sense of first-order stochastic dominance, because the slope of the linear part of $(G^*)^{n-1}$ increases in v^* (see the dashdotted curves in the right panel of Figure 1). Combining this with the fact that such a distribution dominates F if $v^* = \underline{v}$ and is dominated by F if $v^* = \widehat{v}$, it follows that there exists a unique $v^* \in (\underline{v}, \widehat{v})$ that satisfies $G^* \in \text{MPC}(F)$. ■

One immediate implication of Corollary 2.(2) is that consumer surplus strictly increases in n: as n rises, the consumer not only has more options to choose from, but also receives more accurate information about each product.

The final corollary considers the case where F^{n-1} is concave. This is a special case of Corollary 2 and provides a simple sufficient condition under which $(G^*)^{n-1}$ is linear over its support.

Corollary 3 *If F^{n-1} is concave then $(G^*)^{n-1}$ is linear over its support, whose lower bound is $\underline{v}$.*

Example 1 (Truncated exponential distributions) *Suppose $F(v) = \gamma(1 - e^{-\lambda v})$ over $[0, \overline{v}]$ for some $\lambda > 0$, where $\gamma(1 - e^{-\lambda \overline{v}}) = 1$. If $n = 2$, then $F^{n-1} = F$ is concave, in which case $G^* = U[0, 2\mu_F]$ by Corollary 3. If $n \geq \overline{n} \equiv e^{\lambda \overline{v}} + 1 (> 2)$, then F^{n-1} is convex, so $G^* = F$ by Proposition 1.*

[19]This applies whenever F^{n-1} has decreasing density. It also applies if $(F^{n-1})'$ increases only close to $\underline{v}$.

Suppose $n \in (2, \overline{n})$. In this case, F^{n-1} is initially convex and then concave. By Corollary 2, there exists $v^* \in (0, \overline{v})$ such that $G^* = F$ below v^*, while $(G^*)^{n-1}$ is linear above v^*. The cutoff v^* and the upper bound $\overline{v}^*$ of $supp(G^*)$ are implicitly defined by[20]

$$\int_{v^*}^{\overline{v}} v dF(v) = \int_{v^*}^{\overline{v}^*} v dG^*(v) \text{ and } 1 = G^*(\overline{v}^*) = F(v^*)^{n-1} + \left(F(v^*)^{n-1}\right)' (\overline{v}^* - v^*).$$

Specifically, suppose $\lambda = 1$ and $\overline{v} = 3$, as in Figure 1. Then, $\overline{n} \approx 21.09$. Letting $v^*(n)$ denote v^* with n firms, $v^*(3) = 0.3589$, $v^*(7) = 1.5024$, $v^*(14) = 2.4348$, and $v^*(21) = 2.9943$.

5 Competitive Pricing

Notice that the equilibrium distribution G^* in Theorem 1 is independent of the given symmetric price. In this section, we characterize the equilibrium price p^* that corresponds to G^* and use the result to study the effects of strategic advertising on market prices and consumer welfare. We also investigate the equilibrium existence problem by considering a firm's compound deviations.

Given G^*, our problem reduces to the canonical model of Perloff and Salop (1985). Assuming that all other firms charge p^*, an individual firm's optimal pricing problem is given by

$$\max_{p_i} \pi(p_i, G^*, p^*, G^*) = p_i D(p_i, G^*, p^*, G^*) = p_i \int G^*(v_i - p_i + p^*)^{n-1} dG^*(v_i).$$

Deriving the firm's first-order condition and imposing the symmetric equilibrium requirements, we arrive at the following familiar equilibrium pricing formula.

Proposition 2 *In equilibrium, each firm charges*

$$p^* = \frac{1/n}{\int \left(G^*(v)^{n-1}\right)' dG^*(v)}. \tag{3}$$

As usual, this pricing formula shows that the equilibrium price is inversely proportional to the measure of *marginal consumers*, $\int \left(G^*(v)^{n-1}\right)' dG^*(v)$, who are indifferent between a firm's product and the best alternative.[21] Intuitively, when there are more marginal consumers (i.e., it is more likely that the consumer is indifferent between purchasing and not purchasing its product), a firm's marginal benefit from price cutting is larger, leading to a lower p^*.

[20] The first condition is simply due to $G^* \in \text{MPC}(F)$. The second condition follows from $G^*(v^*) = F(v^*)$ and $\left(G^*(v^*)^{n-1}\right)' = \left(F(v^*)^{n-1}\right)'$, which are necessary for G^* to satisfy the conditions in Theorem 1.

[21] Notice that $g^*(v)$ is the density of consumers who assign value v to the firm's product, while $(G^*(v)^{n-1})'$ is the density of those whose best alternative has value v.

To assess the economic effects of strategic advertising, we compare the resulting market outcome to that of the full information benchmark. Let B^* denote the consumer's expected valuation of the purchased product—equivalent to social surplus—and CS^* denote her consumer surplus given G^*. Since the consumer purchases the product with the highest realization among n independent draws from G^*,

$$B^* = \int v dG^*(v)^n \text{ and } CS^* = B^* - p^*. \tag{4}$$

Let p^F, B^F and CS^F denote the corresponding quantities for the full information benchmark.

It is easy to see that $B^* \leq B^F$, because whenever $G^* \neq F$, the consumer selects a suboptimal product with a positive probability. In other words, information loss through strategic advertising causes deadweight loss.

In contrast, the effect of strategic advertising on the market price is unclear. The next example demonstrates that depending on the shape of F, strategic advertising could lower or raise the price. The example further shows that strategic advertising could benefit the consumer—even with the aforementioned deadweight loss—when it induces a significant price drop.[22]

Example 2 (Power distributions) *Consider the duopoly case ($n = 2$) with $F(v) = v^\alpha$ over $[0, 1]$ for some $\alpha > 0$. Clearly, $F(v) = v^\alpha$ is convex if $\alpha \geq 1$ and strictly concave otherwise. By Proposition 1, $G^* = F$ is the unique equilibrium in the former (convex) case. In the latter (concave) case, Corollary 3 implies that $G^*(v) = \beta v$ over $[0, 1/\beta]$ for some $\beta > 0$. Since $G^* \in MPC(F)$ (in particular, $\mathbb{E}_F[x] = \mathbb{E}_{G^*}[v]$), $\beta = (\alpha + 1)/(2\alpha)$. Applying this to (3) and (4) yields*

$$p^F = \frac{2\alpha - 1}{2\alpha^2}, B^F = \frac{2\alpha}{2\alpha + 1}; \ p^* = \frac{\alpha}{\alpha + 1}, B^* = \frac{4\alpha}{3(\alpha + 1)}.$$

It follows that strategic advertising lowers the market price ($p^* < p^F$) if and only if $\alpha > 1/\sqrt{2} \approx 0.7071$; and it benefits consumers ($CS^* > CS^F$) if and only if $\alpha > 0.7928$.

To understand when strategic advertising lowers/raises the market price, we report two results regarding how the equilibrium rice p^F depends on F in the Perloff-Salop model.[23] Whereas the first result is a corollary of a recent theoretical result by Zhou (2017) and Choi et al. (2018), the second result is—to our best knowledge—novel in the literature and of independent interest.

[22] In Appendix F, we consider truncated exponential distributions and show that strategic advertising always lowers the market price, while its effect on consumer surplus varies depending on $\overline{v}$.

[23] Perloff and Salop (1985) already point out that the equilibrium price may or may not increase when F changes in convex order. Their observation does not directly apply to our model, because G^* is a particular type of MPC.

Proposition 3 *(1) The measure of marginal consumers, $\int (F^{n-1})' dF$, decreases if F is proportionally stretched over a larger support.*[24]
(2) Among all distributions whose support belongs to $[\underline{v}, \overline{v}]$, the power distribution $F(v) = \left(\frac{v-\underline{v}}{\overline{v}-\underline{v}}\right)^{2/n}$ minimizes $\int (F^{n-1})' dF$.

As is well-known, preference diversity is the driving force for positive markups in the Perloff-Salop model. One clear way to increase preference diversity is to lengthen the support $[\underline{v}, \overline{v}]$. Proposition 3.(1) confirms that such a change indeed raises the equilibrium price. Unfortunately, such a clear comparative statics result is not available over distributions with the same support. Proposition 3.(2) complements this deficiency by characterizing the price-maximizing distribution among all distributions over $[\underline{v}, \overline{v}]$. The crucial property of the power distribution in Proposition 3.(2) is that marginal consumers are *uniformly distributed* over the support (i.e., the conditional measure of marginal consumers is constant).[25] If marginal consumers are not uniformly distributed, then adjusting F to "flatten" the density of marginal consumers would lead to a smaller measure of marginal consumers.[26]

Proposition 3 enables us to identify two effects of strategic advertising on the market price. On one hand, since $G^* \in \text{MPC}(F)$, strategic advertising yields a value distribution with a narrower support compared to the full information benchmark. Proposition 3.(1) implies that this effect necessarily lowers the equilibrium price. On the other hand, strategic advertising induces a different distribution of marginal consumers, whose effect on the market price is ambiguous in general. Specifically, whenever G^* differs from F and so $(G^*)^{n-1}$ has a locally constant slope of β, the density of marginal consumers $((G^*)^{n-1})' g^* = \beta g^*$ is decreasing, while $(F^{n-1})' f$ can exhibit any given pattern. By Proposition 3.(2), this may or may not increase p^*, depending on the detailed shape of F. This second effect renders the comparison between p^* and p^F ambiguous.

Equilibrium existence. Theorem 1 and Proposition 2 identify the unique candidate for the symmetric pure-price equilibrium: if such an equilibrium exists then it must be as characterized in those two results. However, the existence of symmetric pure-price equilibrium is not guaranteed in our model. This non-existence problem is already present in the Perloff-Salop model (PS, hereafter).

[24] To be precise, we say that $F_2[\underline{v}_2, \overline{v}_2]$ is a proportional stretch of $F_1[\underline{v}_1, \overline{v}_1]$ if $\overline{v}_2 - \underline{v}_2 > \overline{v}_1 - \underline{v}_1$ and

$$F_2(v) = F_1\left(\frac{\overline{v}_1 - \underline{v}_1}{\overline{v}_2 - \underline{v}_2}(v - \underline{v}_2) + \underline{v}_1\right) \text{ for all } v \in [\underline{v}_2, \overline{v}_2].$$

In other words, F_2 is obtained when F_1 is extended to a larger support while keeping its shape. Note that there are no restrictions on the locations of the bounds, that is, it is possible that $\underline{v}_1 > \underline{v}_2$ or $\overline{v}_1 > \overline{v}_2$.

[25] It is easy to show that $(F(v)^{n-1})' f(v) = 4(n-1)(\overline{v} - \underline{v})^2/n^2$ for all $v \in [\underline{v}, \overline{v}]$.

[26] For example, consider the case with $n = 2$, so that the density of marginal consumers is $(F^{n-1})' f = f^2$. Then $\int f^2$ (subject to $\int f = 1$) is minimized when f is independent of v: if $f(v) \neq f(v')$ then one can reduce $f(v)^2 + f(v')^2$ by equally splitting combined density between the two. A similar logic applies to the case with $n > 2$.

However, strategic advertising in our model introduces some novel features into the problem, so that conditions for equilibrium existence are not transferable between the two models.

Consider the case where F^{n-1} is convex, so that $G^* = F$ and $p^* = p^F$. In this case, equilibrium existence in PS is necessary for that in our model, as it prevents profitable *price deviations*. It is not sufficient, however, because there may exist a profitable *compound deviation* in which a firm deviates both from G^* and from p^*. This implies that a pure-price equilibrium may exist only in PS, and our model can produce a different market outcome from PS even if F^{n-1} is convex.

Conversely, it is also possible that a symmetric pure-price equilibrium exists in our model, but not in PS. Recall that if F^{n-1} is not convex then G^* induces a different distribution of marginal consumers from F, generally leading to $p^* \neq p^F$. These changes directly affect a firm's deviation incentives, thereby sometimes sustaining a pure-price equilibrium even when such an equilibrium does not exist in PS. In Appendix E, we offer a specific example for this phenomenon.

Despite these complications, sufficient (but commonly imposed) regularity on F ensures the existence of symmetric pure-price equilibrium in our model, as reported in the following result.

Theorem 2 *If f is log-concave, then there exists a unique symmetric pure-price equilibrium (p^*, G^*) characterized by Theorem 1 and Proposition 2.*

Proof. See Appendix C. ∎

In the Perloff-Salop model, log-concavity of f is sufficient for equilibrium existence: It ensures that a firm's demand function is log-concave, so the first-order condition (3) is sufficient. In our model, it also implies a simple structure of G^*: If f is log-concave then $(G^*)^{n-1}$ has a convex-linear structure as characterized in Corollary 2. This structure preserves log-concavity of a firm's profit function, even after accounting for compound deviations, making the first-order conditions in Theorem 1 and Proposition 2 sufficient for optimality.

6 Discussion

In this section, we examine the robustness of our central economic lessons. Specifically, we consider three economically relevant variations of our baseline model and study whether—and how—our full information result under intense competition depends on each variation.

6.1 Consumer Outside Option

In our baseline model, the consumer does not have an option of not purchasing any product.[27] Normalizing the value of her outside option to zero, the assumption is inconsequential if $\underline{v}-p^* \geq 0$. If $\underline{v} - p^* < 0$, however, it binds and has non-trivial effects on competitive advertising.

Suppose F^{n-1} is convex over its support, so that $G^* = F$ is the unique equilibrium in the baseline advertising game. Now, suppose the consumer has an outside option of value zero, and $p^* \in (\underline{v}, \overline{v})$. In this case, she opts out if the value of the best product is below p^*. This implies that, assuming that all other firms provide full information, firm i's problem is given by

$$\max_{G_i \in \text{MPC}(F)} \int_{p^*}^{\overline{v}} F(v)^{n-1} dG_i(v) = \int 1_{\{v \geq p^*\}} F(v)^{n-1} dG_i(v). \tag{5}$$

The consumer purchases a product when her value for the product exceeds not only her values for the other products ($F(v)^{n-1}$) but also the price ($1_{\{v \geq p^*\}}$). As depicted in the left panel of Figure 2, this introduces a discrete jump for the firm's value function, yielding the following result.

Proposition 4 *If the consumer has a binding outside option (i.e., $p^* \in (\underline{v}, \overline{v})$) then there never exists a full information equilibrium in the advertising game.*

Proof. Because of the structure of $1_{\{v \geq p^*\}} F(v)^{n-1}$, F is dominated by a distribution that pools values around p^* and puts an atom on p^*. Therefore, F cannot be a solution to (5).[28] ■

This result, however, does not exclude the possibility of *asymptotic full information*; that is, it is still possible that the equilibrium distribution G^* converges to F as n tends to infinity. Intuitively, the discrete jump at p^* has the size of $F(p^*)^{n-1}$, which vanishes exponentially fast as n increases; and the limiting value function—which stays equal to 0 and then jumps at $\overline{v}$—is convex. Therefore, if n is sufficiently large then G^* could be similar to F. This convergence indeed holds.

Proposition 5 *In the advertising game with a binding outside option, the equilibrium distribution G^* converges (pointwise) to F as n tends to infinity.*

Note that Proposition 5 requires a full characterization of the equilibrium distribution G^*, which is different from characterizing the solution to (5). The right panel of Figure 2 illustrates the main

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[27] This is a common simplifying assumption in the literature. In the standard Perloff-Salop model, its main advantage is to simplify the equilibrium pricing formula: absent the assumption, the right-hand side in (3) involves p^*, so p^* cannot be expressed in closed form. In our model, together with our focus on pure-price equilibria, it allows us to identify G^* independently of p^*. As we elaborate in Appendix H, if the consumer's outside option is binding then G^* and p^* should be jointly determined, which significantly complicates the technical analysis.

[28] Applying Theorems 1 and 2 of Dworczak and Martini (2019), the unique solution G_i^* to (5) has the following structure: for some $v_1 < p^* < v_2$, $G_i^*(v) = F(v)$ if $v \leq v_1$ or $v \geq v_2$, and G_i puts all remaining probability on p^*. See the dashed brown line in the left panel of Figure 2 for the structure of the solution to (5).

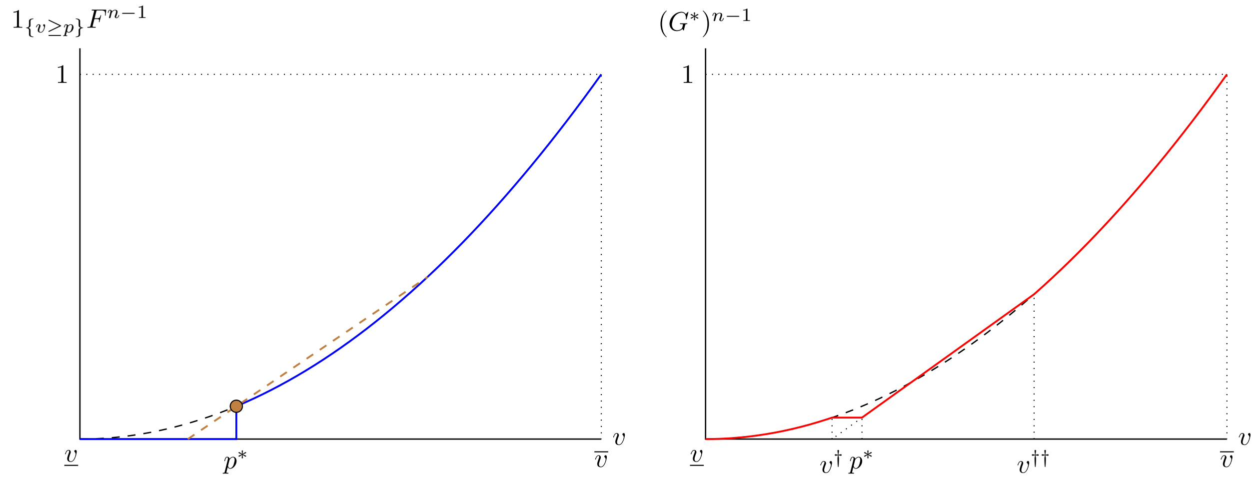

Figure 2: The left panel illustrates why the full information equilibrium does not exist when the consumer has a binding outside option, while the right panel depicts the resulting equilibrium structure. In each panel, the black dashed curve represents the underlying convex distribution.

features of G^*: a firm has no incentive to offer values below p^*, so G^* is flat right below p^*. Meanwhile, by Lemma 2 (which still applies to $[p^*, \overline{v}]$), $(G^*)^{n-1}$ must be locally linear above p^*. Combining these properties with the MPC constraint yields the unique structure of G^*. See Appendix H for a comprehensive equilibrium characterization.

Proposition 5 considers the advertising game with fixed p^*, but n certainly affects the equilibrium price p^* as well. This consideration, however, does not inhibit our convergence result; in fact, it reinforces Proposition 5. The equilibrium price p^* typically decreases in n and necessarily converges to 0 in the limit. This makes the consumer's outside option less binding, thereby facilitating the convergence of G^* to F. To be specific, suppose $\underline{v} = 0$. As n tends to infinity, p^* approaches 0, so the consumer's outside option is not binding in the limit. Proposition 5 provides a stronger result: even if $p^* > 0$ is independent of n, G^* converges to F.

6.2 Multi-unit Demand

In our baseline model, the consumer has a unit demand, purchasing only one out of n products. This means that the firms effectively engage in a winner-take-all contest, whose stark reward structure induces the firms to be particularly aggressive. In this subsection, we study the extent to which our full information result depends on this winner-take-all structure.

We consider the advertising game in which the consumer purchases k (different) products out of n; our baseline model is a special case with $k = 1$. Suppose all other firms play $G \in \text{MPC}(F)$.

If firm i chooses G_i, then its demand is given by

$$D^A(G_i, G) \equiv \int \Phi(v_i; G) dG_i(v_i),$$

where $\Phi_k(v_i; G)$ denotes the probability that v_i exceeds at least $n - k$ draws from G, that is,

$$\Phi(v_i; G) \equiv \sum_{\ell=0}^{k-1} \binom{n-1}{\ell} (1 - G(v_i))^{\ell} G(v_i)^{n-1-\ell}.$$

Clearly, (2) is a special case of $\Phi(v_i; G)$ in which $k = 1$.

For the same reason as given in Section 3, the full information equilibrium exists if and only if $\Phi(\cdot; F)$ is convex over $[\underline{v}, \overline{v}]$. As demonstrated before, this property often holds with single-unit demand: it suffices that F satisfies a certain convexity property or n is sufficiently large. This is no longer the case with multi-unit demand, as formally stated in the following result.

Proposition 6 *In the advertising game with multi-unit demand (i.e., $k > 1$), there never exists a full information equilibrium.*

Proof. By direct calculus, one can show that

$$\Phi'(v; F) = \frac{(n-1) \cdots (n-k)}{(k-1)!} (1 - F(v))^{k-1} F(v)^{n-1-k} f(v). \tag{6}$$

Clearly, $\Phi'(v; F) > 0$ for all $v \in (\underline{v}, \overline{v})$ but whenever $k > 1$, $\Phi'(\overline{v}; F) = 0$. This implies that if $k > 1$ then $\Phi(\cdot; F)$ cannot be convex around $\overline{v}$, regardless of F and n. ■

This result highlights the intuitive fact that multi-unit demand softens competition. Each firm now competes against the k-th best product among the other firms' products (i.e., the k-th order statistic among $(n - 1)$ independent draws), which is less demanding than competing against the first best product. In particular, a firm has no incentive to offer the maximum possible value $\overline{v}$ to the consumer: a slightly lower value would still make it into the consumer's purchase list, so it is more profitable for the firm to pool values around $\overline{v}$.

Proposition 6 suggests that regarding firms' advertising incentives, the intensity of competition cannot be measured by the relative firm size to demand. While firms may provide full information when there are n firms and the consumer has a unit demand, they never do the same when nk firms compete for $k(> 1)$ demands by the consumer. The detail of the demand structure affects firms' advertising incentives.

Despite the above difference, our equilibrium characterization of the advertising game in Section 4 can be readily extended into the current multi-unit case. The only necessary change is to

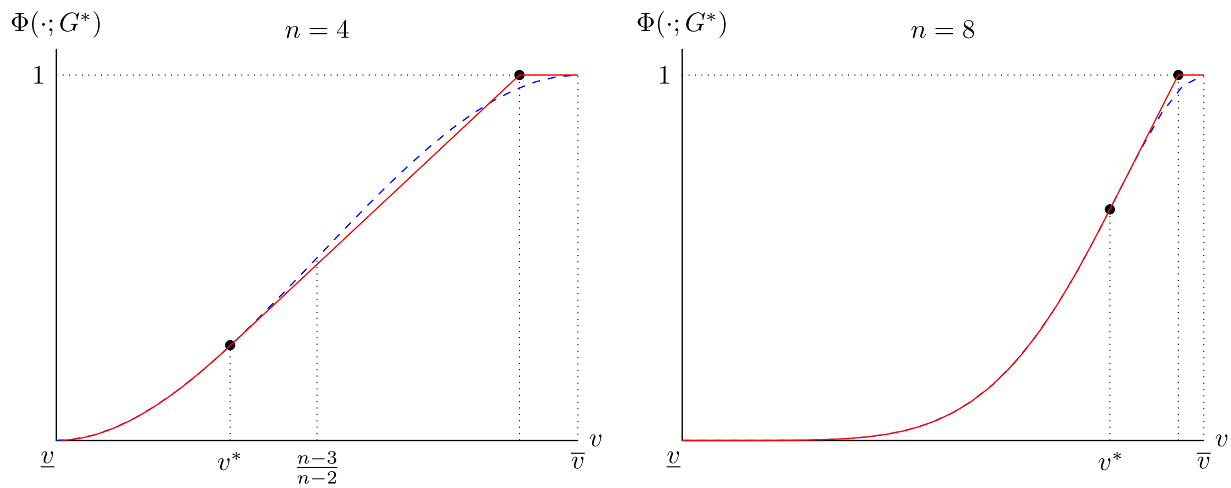

Figure 3: The equilibrium value function $\Phi(\cdot; G^*)$ with multi-unit demand when $F = U[0,1]$ and $k = 2$. In both panels, the dashed curve represents $\Phi(\cdot; F)$.

impose convexity in Lemma 1 and linearity in Lemma 2 on $\Phi(\cdot; G^*)$. We skip a formal analysis—which is a tedious but conceptually straightforward extension of Section 4—and, instead, provide a simple but representative example below.

Example 3 *Suppose $F = U[0,1]$. In the single-unit case, $\Phi(v; F) = F(v)^{n-1} = v^{n-1}$ is convex over $[0,1]$ for any n. Therefore, by Theorem 1, F is the unique equilibrium.*

Suppose $k = 2 < n$. In this case, $\Phi(v; F) = v^{n-1} + (n-1)(1-v)v^{n-2}$, and thus

$$\Phi'(v; F) = (n-1)(n-2)(1-v)v^{n-3}.$$

This expression is single-peaked at $\frac{n-3}{n-2}$, so $\Phi(\cdot; F)$ is convex until $\frac{n-3}{n-2}$ but concave thereafter (see the dashed curves in Figure 3). This implies that, as in Corollary 2 and depicted in Figure 3, there exists $v^* \in \left(0, \frac{n-3}{n-2}\right)$ such that $G^*(v) = F^*(v)$ if $v \le v^*$, and $\Phi(\cdot; G^*)$ is linear above v^*.

Figure 3 suggests that, despite Proposition 7, the equilibrium distribution G^* becomes closer to F as n rises; in particular, v^*—below which $G^* = F$—increases in n. The following result shows that this is indeed the case in general, just as for the case of binding outside option in Section 6.1.

Proposition 7 *In the advertising game with multi-unit demand, the equilibrium distribution G^* converges (pointwise) to F as n tends to infinity.*

Intuitively, if there are infinitely many firms then, even with multi-unit demand, a firm's product can be selected only when it yields the highest possible value $\overline{v}$ to the consumer. As in the unit-demand case, this forces firms to provide precise information when n is large and almost all information in the limit as n tends to infinity.[29]

6.3 Asymmetric Firms

Our clean characterization of competitive advertising owes much to our focus on symmetric environments/equilibria. It is technically far challenging to allow for firm asymmetries.[30] In this subsection, we consider a relatively simple asymmetric case and study how firm asymmetries affect our full information result.

Suppose there are two types of firms, type 1 and type 2.[31] Let n_q denote the number of type-q firms. Each type-q firm is endowed with the distribution F_q over $[\underline{v}_q, \overline{v}_q]$; that is, the consumer's value for a type-q firm's product is drawn from F_q. To avoid triviality, assume that the two supports are not disjoint (i.e., $[\underline{v}_1, \overline{v}_1] \cap [\underline{v}_2, \overline{v}_2] \neq \emptyset$). In addition, without loss of generality, assume $\overline{v}_1 \leq \overline{v}_2$.

We focus on the advertising game, in which each firm maximizes its total demand. Let $H_q(v; G_1, G_2)$ denote the probability that the consumer purchases a firm's product with (net) value v for it. For each $q = 1, 2$, $H_q(v; G_1, G_2)$ is given as follows:

$$H_1(v; G_1, G_2) \equiv G_1(v)^{n_1-1} G_2(v)^{n_2} \text{ and } H_2(v; G_1, G_2) \equiv G_1(v)^{n_1} G_2(v)^{n_2-1}.$$

As before, the full information equilibrium exists if and only if both $H_1(\cdot; F_1, F_2)$ and $H_2(\cdot; F_1, F_2)$ are convex over their support. The following result shows that the property crucially depends on whether F_1 and F_2 have a common upper bound or not.

Proposition 8 *Consider the advertising game with two types of firms. If the supports of F_1 and F_2 have the same upper bound (i.e., $\overline{v}_1 = \overline{v}_2$), then the full information equilibrium exists whenever both $F_1^{n_1-1}$ and $F_2^{n_2-1}$ are convex. Otherwise (i.e., if $\overline{v}_1 < \overline{v}_2$), there never exists a full information equilibrium, regardless of the values of n_1 and n_2.*

[29] For Proposition 7, it is crucial that k is fixed as n increases (or, more generally, k does not increase as fast as n). If k increases at the same rate as n then $\Phi(\cdot; F)$ is not convex—so full information cannot be an equilibrium—even in the limit as $n \to \infty$. As a specific example, consider $k = n - 1$ and $F = U[0, 1]$. In this case, for any n, $\Phi(\cdot; F)$ is concave, so $\Phi(v; G^*) = 1 - (1 - G^*(v))^{n-1}$ should be linear over its support. It can be shown that

$$G^*(v) = 1 - \left(1 - \frac{2(n-1)}{n} v\right)^{n-1} \Leftrightarrow \Phi(v; G^*) = \frac{2(n-1)}{n} v \text{ for } v \in \left[0, \frac{n}{2n(n-1)}\right].$$

In fact, this explains why the full information result does not always hold in the large market model of Ostrovsky and Schwarz (2010): their model can be interpreted as the limit in which both n and k tend to infinity.

[30] It is well known that even in the standard Perloff-Salop model (i.e., when firms' distributions are exogenously given), it is technically hard to analyze the asymmetric case and obtain substantive results. To our knowledge, Quint (2014) provides the most general and comprehensive set of (equilibrium existence and comparative statics) results for the Perloff-Salop framework. His analysis, which utilizes the supermodular-game structure of the model, relies on certain regularity assumptions on the distributions, which do not hold under our equilibrium distributions. In Appendix I, we provide a partial characterization of equilibrium advertising in the duopoly case.

[31] All subsequent arguments can be easily generalized into the case with more than two types.

Proof. The first result follows from the following two facts: (i) if $F_q^{n_q-1}$ is convex then $F_q^{n_q}$ is also convex; and (ii) the product of two non-negative monotone convex functions is also convex. For the second result, observe that

$$\begin{aligned}\lim_{v\to\overline{v}_1-} H_2'(v;F_1,F_2) &= \lim_{v\to\overline{v}_1-}\left(F_1(v)^{n_1}F_2(v)^{n_2-1}\right)' \\ &= n_1F_2(\overline{v}_1)^{n_2-1}f_1(\overline{v}_1)+(n_2-1)F_2(\overline{v}_1)^{n_2-1}f_2(\overline{v}_1) \\ > \lim_{v\to\overline{v}_1+} H_2'(v;F_1,F_2) &= \lim_{v\to\overline{v}_1+}\left(F_2(v)^{n_2-1}\right)' = (n_2-1)F_2(\overline{v}_1)^{n_2-1}f_2(\overline{v}_1).\end{aligned}$$

In other words, $H_2(\cdot;F_1,F_2)$ has a downward kink at $\overline{v}_1$ and so is not convex over $[\underline{v}_2,\overline{v}_2]$, which suffices for the desired result. ■

The economic force behind the full information equilibrium in our baseline symmetric case continues to apply to the current asymmetric case. If there are many competing firms, then a firm can attract the consumer only when she has a particularly high value for its product, so all firms would provide precise information. When $\overline{v}_1=\overline{v}_2$, this "top-down unraveling" logic—always providing the highest value to the consumer—applies equally to all firms. If $\overline{v}_1<\overline{v}_2$, however, the logic does not apply to type-2 firms around $\overline{v}_1$. Due to type-1 firms, they face more competition below $\overline{v}_1$ than above $\overline{v}_1$ and so have an incentive to pool values around $\overline{v}_1$, in order to increase their winning probability against type-1 firms. This incentive is independent of both n_1 and n_2, so the full-informative equilibrium *never* exists in this case.

As for the previous two cases in Sections 6.1 and 6.2, even if $\overline{v}_1<\overline{v}_2$ and so full information can never be an equilibrium, it seems likely that the equilibrium distributions, G_1^* and G_2^*, still converge to F_1 and F_2, as n_1 and n_2 tend to infinity. This is certainly a reasonable conjecture, given that $H_2(\cdot;F_1,F_2)$ is convex in the limit. A formal result, however, requires comprehensive equilibrium characterization, which is technically too challenging and so we leave as an open question.[32]

7 Conclusion

This paper studies an oligopoly model in which the firms compete not only on price but also through informative advertising (i.e., by choosing how much product information to disclose). Unlike most existing studies on informative advertising, but as is common in the recent literature

[32] Given the proof of Proposition 8, one may think that (at least) type-1 firms could provide full information. This is usually not the case; the result holds only when the underlying distributions satisfy some very specific (not usual) properties. For a basic idea, suppose n_1 is sufficiently large, while $n_2=1$. If all type-1 firms provide full information then the (unique) type-2 firm would pool values around $\overline{v}_1$ and put positive mass on $\overline{v}_1$. Unless the type-2 firm can win probability 1, there exists $\tilde{v}(<\overline{v}_1)$ such that $G_2(v)=F_2(v)$ for $v<v^*$ and $G_2(v)$ is constant over $[\tilde{v},\overline{v}_1)$. This makes $H_1(\cdot;F_1,G_2)$ non-convex around $\tilde{v}$: just as in the proof of Proposition 8, $H_1'(\tilde{v}-;F_1,G_2)>H_1'(\tilde{v}+;F_1,G_2)$.

on information design, we impose no structural restriction on feasible advertising content; this enables us to focus on firms' strategic incentives, not subject to any technological constraints, and also obtain a relatively tractable characterization of the equilibrium advertising content. Our main economic insight is that intense competition induces firms to provide precise product information. As elaborated in Section 1, this economic result was also obtained by Ivanov (2013), but our analysis strengthens it in multiple ways. In particular, we show that the result is robust along multiple dimensions. We also demonstrate that strategic advertising, which creates deadweight loss through information loss, has ambiguous implications for both prices and consumer surplus; note that this offers a cautionary tale for policies regarding firms' disclosure requirements.

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A Omitted Proofs

Proof of Proposition 1. The argument before Proposition 1 establishes the sufficiency of convex F^{n-1}. Its necessity follows from Lemma 1: if G^* is an equilibrium in the advertising game then $(G^*)^{n-1}$ must be convex. Therefore, if F^{n-1} is not convex, then F cannot be an equilibrium. For an explicit construction of $G_i \in$ MPC(F) such that $D^A(G_i, F) > D^A(F, F)$ when F^{n-1} is not convex, see the proof of Lemma 1. ■

Proof of Lemma 1. Let $\overline{supp}(G^*)$ denote the convex closure (hull) of $supp(G^*)$, and suppose that $(G^*)^{n-1}$ is not convex over $\overline{supp}(G^*)$. Then, there exist v_1, v_2, v_3, and ε (sufficiently small) such that $v_1 < v_2 < v_3$, $[v_1, v_1 + \varepsilon) \cup (v_3 - \varepsilon, v_3] \subset supp(G^*)$, and

$$G^*(v_2)^{n-1} > \frac{v_3 - v_2}{v_3 - v_1} G^*(v_1)^{n-1} + \frac{v_2 - v_1}{v_3 - v_1} G^*(v_3)^{n-1}. \tag{7}$$

Let $\delta \in (0, \varepsilon)$ and $\delta' \in (0, \varepsilon)$ be the values such that

$$v_2 = \mathbb{E}_{G^*}[v | v \in [v_1, v_1 + \delta) \cup (v_3 - \delta', v_3]] = \frac{\int_{v_1}^{v_1+\delta} v dG^*(v) + \int_{v_3-\delta'}^{v_3} v dG^*(v)}{\Delta + \Delta'},$$

where $\Delta \equiv \int_{v_1}^{v_1+\delta} dG^*(v)$ and $\Delta' \equiv \int_{v_3-\delta'}^{v_3} dG^*(v)$. The pair exists because $\mathbb{E}_{G^*}[v|v \in [v_1, v_1 + \delta) \cup (v_3 - \delta', v_3]]$ can take any value in (v_1, v_3), depending on the relative weights of δ and δ'.

Consider the following alternative distribution function G_i:

$$G_i(v) = \begin{cases} G^*(v) & \text{if } v < v_1, \\ G^*(v_1) & \text{if } v \in [v_1, v_1 + \delta), \\ G^*(v) - \Delta & \text{if } v \in [v_1 + \delta, v_2), \\ G^*(v) + \Delta' & \text{if } v \in [v_2, v_3 - \delta'), \\ G^*(v_3) & \text{if } v \in [v_3 - \delta', v_3), \\ G^*(v) & \text{if } v \geq v_3. \end{cases}$$

In other words, G_i differs from G^* only in that it takes probability mass from $[v_1, v_1 + \delta) \cup (v_3 - \delta', v_3]$ and assigns it to v_2. By construction, $G_i \in \text{MPC}(G^*) \subset \text{MPC}(F)$. In addition,

$$\int (G^*)^{n-1} dG_i - \int (G^*)^{n-1} dG^* = (\Delta + \Delta') G^*(v_2)^{n-1} - \left(\int_{v_1}^{v_1+\delta} (G^*)^{n-1} dG^* + \int_{v_3-\delta'}^{v_3} (G^*)^{n-1} dG^* \right),$$

which is strictly positive for ε sufficiently small, due to (7). This violates the equilibrium requirement that $D^A(G^*, G^*) \geq D^A(G_i, G^*)$, establishing that $(G^*)^{n-1}$ being convex over $\overline{supp}(G^*)$ is necessary for G^* to be an equilibrium.

Since $(G^*)^{n-1}$ must have non-decreasing density over $\overline{supp}(G^*)$, $supp(G^*) = \overline{supp}(G^*)$. ■

Proof of Lemma 2. Define a function $W : [\underline{v}, \overline{v}] \to \mathcal{R}$ as follows:

$$W(v) = \int_{\underline{v}}^{v} (F(v) - G^*(v)) dv.$$

Since $G^* \in \text{MPC}(F)$, $W(v) \geq 0$ for any v and $W(\overline{v}) = 0$ (see Section 6.D in Mas-Colell et al., 1995). We show that whenever $W(v_1) = W(v_2) = 0$ and $W(v) > 0$ for all $v \in (v_1, v_2)$, $(G^*)^{n-1}$ must be linear over $[v_1, \min\{v_2, \overline{v}^*\}]$, where $\overline{v}^*$ denotes the upper bound of $supp(G^*)$. This suffices for Lemma 2, because $W(v) = 0$ only when $F(v) = G(v)$, and $W(v) = 0$ if and only if F and G^* have the same mean over $[\underline{v}, v]$.

Observe that the continuity of G^* (from Lemma 1) implies that G^* must coincide with F at v_1 and v_2. We consider the following two cases.

(i) Suppose that $v_2 \leq \overline{v}^*$. Let G_i denote the distribution that coincides with F on $[v_1, v_2]$ and follows G^* elsewhere; that is, $G_i(v) = F(v)$ if $v \in [v_1, v_2]$ and $G_i(v) = G^*(v)$ otherwise. By construction, G_i is an MPC of F (because G^* is an MPC of F over $[\underline{v}, v_1]$ and $[v_2, \overline{v}]$), while it is a mean-preserving spread of G^* (because G^* is an MPC of F over $[v_1, v_2]$).

Since G^* must be a best response to itself, we have

$$\int G^*(v)^{n-1} dG_i(v) \leq \int G^*(v)^{n-1} dG^*(v) \Leftrightarrow \int_{v_1}^{v_2} G^*(v)^{n-1} dG_i(v) \leq \int_{v_1}^{v_2} G^*(v)^{n-1} dG^*(v).$$

On the other hand, since $(G^*)^{n-1}$ is convex over (v_1, v_2) and $G^* \in \text{MPC}(G_i)$, we also have

$$\int G^*(v)^{n-1} dG_i(v) \geq \int G^*(v)^{n-1} dG^*(v) \Leftrightarrow \int_{v_1}^{v_2} G^*(v)^{n-1} dG_i(v) \geq \int_{v_1}^{v_2} G^*(v)^{n-1} dG^*(v),$$

leading to $\int_{v_1}^{v_2} G^*(v)^{n-1} dG_i(v) = \int_{v_1}^{v_2} G^*(v)^{n-1} dG^*(v)$. Since $G_i = F \neq G^*$ on (v_1, v_2) and $(G^*)^{n-1}$ is convex, this equality holds only when $(G^*)^{n-1}$ is linear over (v_1, v_2).

(ii) Suppose that $v_2 > \overline{v}^*$. Since $G^* \in \text{MPC}(F)$, this case arises only when $v_2 = \overline{v}$ (see Figure 1). The previous proof does not directly apply here, because $(G^*)^{n-1}$ is convex only over $[v_1, \overline{v}^*] \subsetneq [v_1, v_2]$. To fix this problem, let $\tilde{v}$ be the value such that

$$\mathbb{E}_F[v | v \geq \tilde{v}] \equiv \int_{\tilde{v}}^{\overline{v}} v \frac{dF(v)}{1 - F(\tilde{v})} = \overline{v}^*.$$

This value is well defined, because $\mathbb{E}_F[v|v \geq \tilde{v}]$ is strictly increasing in $\tilde{v}$, $\mathbb{E}_F[v|v \geq v_1] = \mathbb{E}_{G^*}[v|v \geq v_1] < \overline{v}^*$, and $\mathbb{E}_F[v|v \geq \overline{v}^*] > \overline{v}^*$.

Let G_i denote the distribution that coincides with G^* below v_1, follows F until $\tilde{v}$, and then places all remaining mass on $\overline{v}^*$. Formally,

$$G_i(v) = \begin{cases} G^*(v) & \text{if } v < v_1, \\ \min\{F(v), F(\tilde{v})\} & \text{if } v \in [v_1, \overline{v}^*), \\ 1 & \text{if } v \geq \overline{v}^*. \end{cases}$$

By construction, $G_i \in \text{MPC}(F)$, while $G^* \in \text{MPC}(G_i)$: this latter result holds because $\mathbb{E}_{G_i}[v|v \geq v_1] = \mathbb{E}_F[v|v \geq v_1] = \mathbb{E}_{G^*}[v|v \geq v_1]$ and, by the definition of G_i and the fact that G^* is an MPC of F over $[v_1, v_2]$, G_i crosses G^* only once from above. Given this, a similar proof to the previous one applies: by the optimality of G^* over G_i and the convexity of $(G^*)^{n-1}$ over $[v_1, \overline{v}^*]$,

$$\int_{v_1}^{\overline{v}^*} G^*(v)^{n-1} dG_i(v) = \int_{v_1}^{\overline{v}^*} G^*(v)^{n-1} dG^*(v).$$

Since $G^* \neq G_i$ over $[v_1, \overline{v}^*]$, this equality holds only when $(G^*)^{n-1}$ is linear over $(v_1, \overline{v}^*)$. ■

Proof of Sufficiency in Theorem 1. Suppose that G^* is an MPC of F that satisfies Lemmas 1 and 2.[33] It is an equilibrium in the advertising game if and only if it is a solution to

$$\max_{G_i \in \text{MPC}(F)} D^A(G_i, G^*) = \int G^*(v)^{n-1} dG_i(v). \tag{8}$$

[33] See Appendix B for our construction of G^* and the uniqueness proof.

We show that G^* solves this programming problem by applying the following theorem.

Theorem 3 *(Dworczak and Martini, 2019) Consider the following programming problem, where F is a distribution over $[\underline{v}, \overline{v}]$:*

$$\max_{G \in MPC(F)} \int_{\underline{v}}^{\overline{v}} u(x) dG(x).$$

A distribution $G^* \in MPC(F)$ solves the above problem if there exists a convex function $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}$ such that (i) $\phi(x) \geq u(x)$ for all $x \in [\underline{v}, \overline{v}]$, (ii) $supp(G^*) \subset \{x \in [\underline{v}, \overline{v}] : u(x) = \phi(x)\}$, and (iii) $\int_{\underline{v}}^{\overline{v}} \phi(x) dG^*(x) = \int_{\underline{v}}^{\overline{v}} \phi(x) dF(x)$.

In what follows, we construct a function ϕ that satisfies all three properties in the theorem. Let $\underline{v}^*$ and $\overline{v}^*$ denote the lower and upper bounds of $supp(G^*)$, respectively. By Lemma 1, $supp(G^*) = [\underline{v}^*, \overline{v}^*]$ and, since $G^* \in \text{MPC}(F)$, $[\underline{v}^*, \overline{v}^*] \subseteq [\underline{v}, \overline{v}]$. For each $v \in [\underline{v}, \overline{v}]$, let

$$\phi(v) \equiv \begin{cases} G^*(v)^{n-1} & \text{if } v \in [\underline{v}, \overline{v}^*], \\ \alpha(v - \overline{v}^*) + 1 & \text{if } v \in (\overline{v}^*, \overline{v}], \end{cases}$$

where

$$\alpha \equiv \limsup_{v \to \overline{v}^* -} \frac{G^*(\overline{v}^*)^{n-1} - G^*(v)^{n-1}}{\overline{v}^* - v}. \tag{9}$$

If $\overline{v}^* = \overline{v}$ then $\phi(v) = G^*(v)^{n-1}$ over $[\underline{v}, \overline{v}]$. If $\overline{v}^* < \overline{v}$, then $G(\overline{v}^*) = 1 > F(\overline{v}^*)$, in which case, by Lemma 2, $(G^*)^{n-1}$ must be linear right below $\overline{v}^*$ and $\phi(v)$ linearly extends $(G^*)^{n-1}$ above $\overline{v}^*$.

By construction, $\phi(v) \geq G^*(v)^{n-1}$ for all $v \in [\underline{v}, \overline{v}]$, and $\phi(v) = G^*(v)^{n-1}$ for all $v \in [\underline{v}^*, \overline{v}^*]$. Therefore, (i) and (ii) in Theorem 3 hold. For (iii), notice that Lemma 2 and our construction of $\phi(v)$ imply that ϕ can be described by a partition $\{v_0 = \underline{v}, v_1, ..., v_m = \overline{v}\}$ such that for each $k = 1, ..., m$, $G^*(v) = F(v)$ over $[v_{k-1}, v_k]$, or ϕ is linear over $[v_{k-1}, v_k]$ and G^* is an MPC of F over $[v_{k-1}, v_k]$. Either case, we have

$$\int_{v_{k-1}}^{v_k} \phi(v) dG^*(v) = \int_{v_{k-1}}^{v_k} \phi(v) dF(v), \text{ leading to } \int_{\underline{v}}^{\overline{v}} \phi(v) dG^*(v) = \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v).$$

■

Proof of Corollary 2.(2). For each $n \geq 2$, let G_n^* denote the corresponding equilibrium distribution and v_n^* denote the value of v^* over which $(G_n^*)^{n-1}$ is linear. Fix $k \geq 2$. The problem is trivial if $v_k^* = \underline{v}$ or $v_{k+1}^* = \overline{v}$, so assume that $v_k^* > \underline{v}$ and $v_{k+1}^* < \overline{v}$. Given the proof for (1) in the main text, it suffices to show that if the cutoff v^* remains constant as n increases from k to $k+1$, then the resulting distribution, denoted by $\tilde{G}$, requires a higher mean than F. Here, we prove a stronger result, namely, that $\tilde{G}$ first-order stochastically dominates $G_k^* \in \text{MPC}(F)$.

For notational simplicity, let $a \equiv F(v_k^*)$ and $b \equiv f(v_k^*)(v - v_k^*)/F(v_k^*)$. Then,

$$G_k^*(v) = \begin{cases} F(v) & \text{if } v \le v_k^* \\ \min\left\{\left(a^{k-1} + (k-1)a^{k-1}b\right)^{1/(k-1)}, 1\right\} & \text{if } v > v_k^*, \end{cases} \tag{10}$$

while

$$\tilde{G}(v) = \begin{cases} F(v) & \text{if } v \le v_k^* \\ \min\left\{\left(a^k + ka^k b\right)^{1/k}, 1\right\} & \text{if } v > v_k^*. \end{cases}$$

We show that for any $v > v_k^*$ such that $G_k^*(v) \le 1$,

$$G_k^*(v) = \left(a^{k-1} + (k-1)a^{k-1}b\right)^{\frac{1}{k-1}} > \tilde{G}(v) = \left(a^k + ka^k b\right)^{\frac{1}{k}},$$

which is equivalent to

$$(1 + (k-1)b)^k > (1 + kb)^{k-1} \Leftrightarrow (1 + kb)\left(\frac{1 + (k-1)b}{1 + kb}\right)^{k-1} > 1.$$

The desired result follows from the fact that the left-hand side in the final expression is equal to 1 if $b = 0$ (i.e., $v = v_k^*$) and strictly increasing in b (i.e., in v).

It remains to prove that $G_k^* \in \text{MPC}(G_{k+1}^*)$. The problem is trivial if $v_k^* = \overline{v}$ (i.e., $G_k^* = F$). There are the following two cases to consider.

Case 1: $v_k^* < v_{k+1}^*$. By their definitions, $G_k^*(v) = G_{k+1}^*(v) = F(v)$ for $v \le v_k^*$, and $G_{k+1}^*(v) = F(v)$ over $[v_k^*, v_{k+1}^*]$. Combining this with $G_k^* \in \text{MPC}(F)$ implies that there exists $\varepsilon > 0$ such that $G_{k+1}^*(v) > G_k^*(v)$ whenever $v \in (v_k^*, v_k^* + \varepsilon)$. Since $G_k^*, G_{k+1}^* \in \text{MPC}(F)$, it suffices to show that G_{k+1}^* crosses G_k^* once from above.

Suppose to the contrary that G_{k+1}^* crosses G_k^* more than once. Let $\hat{v}$ be the smallest value at which G_{k+1}^* crosses G_k^* from above. Recall that $(G_k^*)^{k-1}$ is linear above v_k^*. Furthermore, since $(G_{k+1}^*)^k$ is linear above v_{k+1}^*, $(G_{k+1}^*)^{k-1} = \left((G_{k+1}^*)^k\right)^{(k-1)/k}$ is strictly concave above v_{k+1}^*. Therefore, it must be that $\hat{v} \in (v_k^*, v_{k+1}^*)$; otherwise, G_{k+1}^* crosses G_k^* only once. Then, F^{k-1} must be concave at some $\tilde{v} \le \hat{v}$. Since F^{k-1} has quasi-concave density, F^{k-1} is concave for all $v \ge \tilde{v}$. However, this implies that G_{k+1}^* concave for all $v \ge \tilde{v}$ as G_{k+1}^* is smooth at v_{k+1}^*, and thus G_{k+1}^* does not cross G_k^* for all $v > \hat{v}$, leading to a contradiction.

Case 2: $v_k^* = v_{k+1}^*$. The above argument implies that this case occurs only when $v_k^* = v_{k+1}^* = \underline{v}$. Then Corollary 3 implies that $(G_k^*)^{k-1}$ and $(G_{k+1}^*)^k$ are linear over $[\underline{v}, \overline{v}_k^*]$ and $[\underline{v}, \overline{v}_{k+1}^*]$, respectively, where $\overline{v}_k^*$ and $\overline{v}_{k+1}^*$ are determined by the condition that $G_k^*, G_{k+1}^* \in \text{MPC}(F)$. Similarly to Case 1, $(G_k^*)^{k-1}$ is linear, while $(G_{k+1}^*)^{k-1}$ is strictly concave (below 1). Combining this with the fact that G_k^* and G_{k+1}^* have the same mean, it follows that G_{k+1}^* crosses G_k^* exactly once from above, which suffices for $G_k^* \in \text{MPC}(G_{k+1}^*)$. ■

Proof of Proposition 3. (1) Suppose F_2 is a proportional stretch of F_1, as defined in Footnote 24. By a recent result by Zhou (2017) and Choi, Dai, and Kim (2018), it suffices to show that F_2 dominates F_1 in dispersive order; that is, for any $0 < a < b < 1$

$$F_2^{-1}(b) - F_2^{-1}(a) \geq F_1^{-1}(b) - F_1^{-1}(a).$$

Let $x_{1,a} \equiv F_1^{-1}(a)$, $x_{1,b} \equiv F_1^{-1}(b)$, $x_{2,a} \equiv F_2^{-1}(a)$, and $x_{2,b} \equiv F_2^{-1}(b)$. Since F_2 is a proportional stretch of F_1, we have

$$F_1(x_{1,a}) = a = F_2(x_{2,a}) = F_1\left(\frac{\overline{v}_1 - \underline{v}_1}{\overline{v}_2 - \underline{v}_2}(x_{2,a} - \underline{v}_2) + \underline{v}_1\right) \Rightarrow x_{1,a} = \frac{\overline{v}_1 - \underline{v}_1}{\overline{v}_2 - \underline{v}_2}(x_{2,a} - \underline{v}_2) + \underline{v}_1.$$

Similarly, we have

$$x_{1,b} = \frac{\overline{v}_1 - \underline{v}_1}{\overline{v}_2 - \underline{v}_2}(x_{2,b} - \underline{v}_2) + \underline{v}_1.$$

It then follows that

$$F_1^{-1}(b) - F_1^{-1}(a) = x_{1,b} - x_{1,a} = \frac{\overline{v}_1 - \underline{v}_1}{\overline{v}_2 - \underline{v}_2}(x_{2,b} - x_{2,a}) = \frac{\overline{v}_1 - \underline{v}_1}{\overline{v}_2 - \underline{v}_2}\left(F_2^{-1}(b) - F_2^{-1}(a)\right) \leq F_2^{-1}(b) - F_2^{-1}(a),$$

where the last inequality holds because $\overline{v}_1 - \underline{v}_1 \leq \overline{v}_2 - \underline{v}_2$.

(2) Let $\overline{\mathcal{F}}$ denote the set of all distributions defined over $[\underline{v}, \overline{v}]$ and $\mathcal{F}$ denote the subset of $\overline{\mathcal{F}}$ that includes only those distributions with piecewise continuous density. Any distribution $F \in \overline{\mathcal{F}}/\mathcal{F}$ can be approximated by those in $\mathcal{F}$, so we focus on $\mathcal{F}$. Then, the problem of minimizing $\int (F^{n-1})' dF$ can be written as

$$\max_{f(\cdot)} - \int_{\underline{v}}^{\overline{v}} (n-1)F(v)^{n-2} f(v)^2 dv \text{ s.t. } \dot{F}(v) = f(v) \geq 0, F(\underline{v}) = 0, \text{ and } F(\overline{v}) = 1.$$

The corresponding Hamiltonian is given by $H = -(n-1)F^{n-2}f^2 + \xi f$.

First, consider the case when $n = 2$. Since the Hamiltonian is simply $H = -f^2 + \xi f$, the optimal solution f and the corresponding costate variable ξ satisfy

$$\frac{\partial H}{\partial f} = -2f + \xi \leq 0, = 0 \text{ if } f > 0, \text{ and } \dot{\xi} = 0.$$

It is easy to see that $\xi(v) = \xi_0$ and $f(v) = \xi_0/2$ for some ξ_0. The value of ξ_0 is determined from the boundary conditions:

$$1 = F(\overline{v}) = \int_{\underline{v}}^{\overline{v}} f(v)dv = \frac{(\overline{v} - \underline{v})\xi_0}{2} \Rightarrow \xi_0 = \frac{2}{\overline{v} - \underline{v}} > 0.$$

It follows that $F(v) = \frac{v - \underline{v}}{\overline{v} - \underline{v}}$ uniquely satisfies Pontryagin's maximum principle.

Now consider the case when $n > 2$. In this case, the Hamiltonian is $H = -(n-1)F^{n-2}f^2 + \xi f$, so the optimal solution f and the corresponding costate variable ξ satisfy

$$\frac{\partial H}{\partial f} = -2(n-1)F^{n-2}f + \xi(v) \leq 0, = 0 \text{ if } f > 0 \text{ and}$$

$$\dot{\xi} = -\frac{\partial H}{\partial F} = (n-1)(n-2)F^{n-3}f^2.$$

The second condition can be rewritten as $\xi(v) = (n-1)\int_{\underline{v}}^{v}(n-2)F(x)^{n-3}f(x)^2 dx + c$ for some constant c. Combining this with the first condition leads to

$$2(n-1)F(v)^{n-2}f(v) - (n-1)\int_{\underline{v}}^{v}(n-2)F(x)^{n-3}f(x)^2 dx = c.$$

This equation should hold for any v, so

$$\begin{aligned} 0 &= \frac{d}{dv}\left(2(n-1)F(v)^{n-2}f(v) - (n-1)\int_{\underline{v}}^{v}(n-2)F(x)^{n-3}f(x)^2 dx\right) \\ &= (n-1)(n-2)F(v)^{n-3}f(v)^2 + 2F(v)^{n-2}f'(v) = (n-1)\left(F(v)^{n-2}f(v)^2\right)'. \end{aligned}$$

This means that at the optimal solution, $F(v)^{n-2}f(v)^2$ should be independent of v. It can be directly shown that the given power function is the unique distribution over $[\underline{v}, \overline{v}]$ that possesses the property.

Note that $-F^{n-2}f^2$ is concave in (F, f) and, whether $n = 2$ or $n > 2$, $\xi \geq 0$. This ensures that Pontryagin's maximum principle is not only necessary but also sufficient (see, e.g., Theorem 4.6.2. in Leonard and Van Long, 1992) so the distribution characterized above is indeed an optimal solution. ■

Proof of Proposition 5. Recall that we consider the case where n is so large that F^{n-1} is convex over its support. By Proposition 9 in Appendix H, for any (large) n, there exist $v_n^{\dagger}$ and $v_n^{\dagger\dagger}$ such that

$v_n^\dagger < p^* < v^{\dagger\dagger}$ and G_n^* is given by

$$G_n^*(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \le v_n^\dagger, \\ F(v_n^\dagger)^{n-1}, & \text{if } v \in (v_n^\dagger, p^*], \\ F(v_n^\dagger)^{n-1} + \beta_n(v - p^*), & \text{if } v \in (p^*, v_n^{\dagger\dagger}], \\ F(v)^{n-1}, & \text{if } v > v_n^{\dagger\dagger}, \end{cases}$$

where

$$F(v_n^\dagger)^{n-1} + \beta_n(v_n^\dagger - p^*) = 0 \Leftrightarrow \beta_n \equiv \frac{F(v_n^\dagger)^{n-1}}{p^* - v_n^\dagger}.$$

Clearly, G_n^* should be an MPC of F over $[v_n^\dagger, v_n^{\dagger\dagger}]$.

Given the above characterization, it suffices to show that $v_n^\dagger$ approaches p^* as $n \to \infty$. Suppose, to the contrary, there exist $\varepsilon > 0$ and a subsequence $\{v_{n_k}^\dagger\}$ such that $p^* - v_{n_k}^\dagger > \varepsilon$ for all k. Then

$$\frac{(F(p^*)^{n_k-1})'}{\beta_{n_k}} = \frac{(n_k-1)f(p^*)F(p^*)^{n_k-2}}{F(v_{n_k}^\dagger)^{n_k-1}/(p^* - v_{n_k}^\dagger)} = \frac{f(p^*)(n_k-1)(p^* - v_{n_k}^\dagger)}{F(p^*)}\left(\frac{F(p^*)}{F(v_{n_k}^\dagger)}\right)^{n_k-1} > 1,$$

for sufficiently large k. However, this implies that $F(v_{n_k}^\dagger)^{n_k-1} + \beta_{n_k}(v - p^*)$ stays uniformly below $F(v)^{n_k-1}$ over $[\underline{v}, \overline{v}]$, which contradicts the requirement that $G_{n_k}^*$ is an MPC of F over $[v_{n_k}^\dagger, v_{n_k}^{\dagger\dagger}]$. ■

Proof of Proposition 7. For each n, let v_n^* be the first point at which $\Phi(\cdot; G^*)$ is locally linear:

$$v_n^* \equiv \min\{v \in [\underline{v}, \overline{v}] : \text{there exists } \varepsilon > 0 \text{ s.t. } \Phi(\cdot; G^*) \text{ is linear over } [v, v+\varepsilon]\}.$$

For each n, let $\overline{v}_n^*$ denote the maximal value such that $\Phi(\cdot; G^*)$ is linear over $[v_n^*, \overline{v}_n^*]$. For the same reason as in our baseline model, G^* must be an MPC of F over $[v_n^*, \overline{v}_n^*]$. Note that this implies that $\Phi(\cdot; G^*)$ must cross $\Phi(\cdot; F)$ at least once from below over the interval, as the same property should hold between G^* and F.

Since $G^*(v) = F(v)$ for all $v \le v_n^*$, it suffices to show that v_n^* approaches $\overline{v}$ as n tends to ∞. Toward a contradiction, suppose there is an infinite subsequence $\{n_k\}$ such that $v_{n_k}^* \le v^\dagger$ for all n_k and some $v^\dagger < \overline{v}$. We show that if n_k is sufficiently large then $\Phi(\cdot; G^*)$ stays uniformly below $\Phi(\cdot; F)$, and thus G^* cannot be an MPC of F, which is a contradiction.

We first show that for n sufficiently large, $\Phi(\cdot; F)$ is convex over $[\underline{v}, v^\dagger]$. From (6), we get

$$\begin{aligned}\Phi''(v; F) \;=\; & \frac{(n-1)\cdots(n-k)}{(k-1)!}(1-F(v))^{k-2}F(v)^{n-2-k}f(v)^2 \\ & \times\left((n-1-k)(1-F(v))f(v)^2 - (k-1)F(v)f(v)^2 + (1-F(v))F(v)f'(v)\right).\end{aligned}$$

To show that $\Phi''(v;F) > 0$ for all $v \in [\underline{v}, v^\dagger]$, let $\varepsilon \equiv \min\{f(v) : v \in [\underline{v}, \overline{v}]\}$ and $M \equiv \max\{|f'(v)| : v \in [\underline{v}, \overline{v}]\}$ (as in the proof of Corollary 1). Then, we have

$$\begin{aligned} & (n-1-k)(1-F(v))f(v)^2 - (k-1)F(v)f(v)^2 + (1-F(v))F(v)f'(v) \\ > \;& (n-1-k)(1-F(v^\dagger))\varepsilon^2 - (k-1)\varepsilon^2 - M \end{aligned}$$

which is positive for n sufficiently large; note that for this argument, it is crucial that $v^\dagger < \overline{v}$, so $F(v^\dagger) < 1$.

Fix any sufficiently large n_k. Then, by the equilibrium structure, and because $\Phi(\cdot;F)$ is convex until $v^\dagger$, it must be that for all $v \in [v^*_{n_k}, \overline{v}^*_{n_k}]$,

$$\Phi(v;G^*) = \Phi(v^*_{n_k};F) + \Phi'(v^*_{n_k};F)(v - v^*_{n_k}) \leq \begin{cases} \Phi(v;F) & \text{if } v \in [v^*_{n_k}, v^\dagger] \\ \Phi(v^\dagger;F) + \Phi'(v^\dagger;F)(v - v^\dagger) & \text{if } v \in (v^\dagger, \overline{v}^*_{n_k}]. \end{cases}$$

From (6),

$$\Phi'(v^\dagger;F) = \frac{(n-1)\cdots(n-k)}{(k-1)!}(1-F(v^\dagger))^{k-1}F(v^\dagger)^{n-1-k}f(v^\dagger)$$

which converges to 0 as n tends to ∞, because $(n-1)\cdots(n-k)$ increases polynomially fast, while $F(v^\dagger)^{n-1-k}$ decreases exponentially fast. Combining this with the above arguments, it follows that $\Phi(v;G^*)$ stays uniformly below $\Phi(v;F)$ over $[v^*_{n_k}, \overline{v}]$, which is a contradiction. ■

B Existence and Uniqueness of G^*

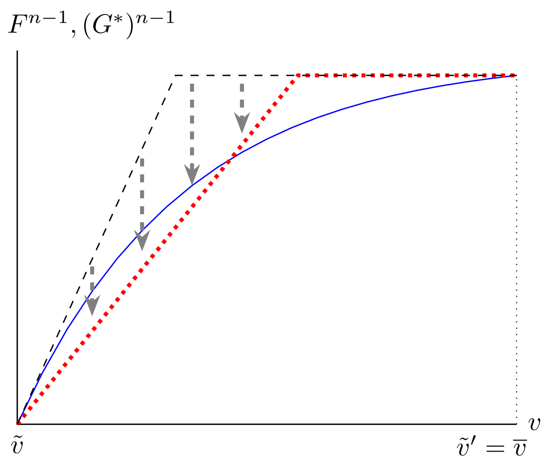
This appendix provides a formal proof for the second statement in Theorem 1, namely, that given F, there exists a unique $G^* \in \text{MPC}(F)$ that satisfies Lemmas 1 and 2.

Preliminaries for existence. For each $\tilde{v} \in [\underline{v}, \overline{v}]$ and $a \in \mathcal{R}_+$, let $H_{\tilde{v},a}$ denote the distribution such that $H_{\tilde{v},a}(v) = F(v)$ if $v \leq \tilde{v}$ and $H^{n-1}_{\tilde{v},a}$ is linear above $\tilde{v}$ with slope a. Formally,

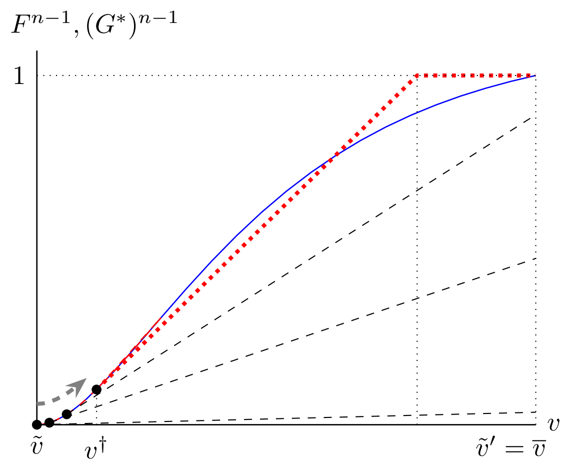
$$H_{\tilde{v},a}(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \leq \tilde{v}, \\ \min\{a(v-\tilde{v}) + F(\tilde{v})^{n-1}, 1\}, & \text{if } v \in (\tilde{v}, \overline{v}), \\ 1, & \text{if } v = \overline{v}. \end{cases}$$

For examples, see the dashed and dotted lines in Figure 4. Given $H_{\tilde{v},a}$, define a function $W_{\tilde{v},a} : [\underline{v}, \overline{v}] \to \mathcal{R}$ as follows:

$$W_{\tilde{v},a}(v) \equiv \int_{\underline{v}}^{v} (F(x) - H_{\tilde{v},a}(x))dx.$$



An illustration of Lemma 3



An illustration of Lemma 4

Figure 4: This figure visualizes the logic behind Lemmas 3 and 4. In both panels, $\tilde{v} = \underline{v} = 0$, $\overline{v} = 3$, and $F(v) = (1 - e^{-v})/(1 - e^{-\overline{v}})$. But, in the left panel, $n = 2$, while $n = 3$ in the right panel.

As is well known, $H_{\tilde{v},a}$ is an MPC of F over $[v_1, v_2]$ if and only if $W_{\tilde{v},a}(v_1) = W_{\tilde{v},a}(v_2) = 0$ and $W_{\tilde{v},a}(v) \geq 0$ for all $v \in [v_1, v_2]$. We use this result to check the MPC constraint for G^*.

Consider a distribution $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$. By construction, $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) = 0$ for all $v \leq \tilde{v}$. For $v > \tilde{v}$, there are two cases: (i) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) \leq 0$ for some $v \in (\tilde{v}, \overline{v}]$, and (ii) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v}]$. The following lemmas illustrate how to proceed the construction in each case.

Lemma 3 *If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) \leq 0$ for some $v \in (\tilde{v}, \overline{v})$, then there exist $a^* \in (0, (F(\tilde{v})^{n-1})']$ and $\tilde{v}' \in (\tilde{v}, \overline{v}]$ such that $H_{\tilde{v},a^*}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$.*

Proof. Suppose that there exists $v \in (\tilde{v}, \overline{v})$ such that $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) \leq 0$. If $a = 0$, then F stays uniformly above $H_{\tilde{v},a}$, so $W_{\tilde{v},a}(v) > 0$ for all v. In addition, for each $v \in (\tilde{v}, \overline{v}]$, $W_{\tilde{v},a}(v)$ is continuously and strictly increasing in a. Therefore, there always exists $a^* \in (0, (F(\tilde{v})^{n-1})')$ and $\overline{v}' \in (\tilde{v}, \overline{v}]$ such that $W_{\tilde{v},a^*}(v) \geq 0$ for all $v \in (\tilde{v}, \tilde{v}')$ and $W_{\tilde{v},a^*}(\tilde{v}') = 0$. It follows that $H_{\tilde{v},a^*}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$. ■

Lemma 4 *If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v})$, then either F^{n-1} is convex over $[\tilde{v}, \overline{v}]$, or there exist $v^\dagger, \tilde{v}' \in (\tilde{v}, \overline{v}]$ such that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$.*

Proof. Suppose that F^{n-1} is convex over $[\tilde{v}, \overline{v}]$. In this case, for any $v^\dagger \in [\tilde{v}, \overline{v}]$, $H_{v^\dagger,(F(v^\dagger)^{n-1})'}(v) < F(v)$ for all $v \in (\tilde{v}, \overline{v})$, so it cannot be an MPC of F. Now suppose that F^{n-1} is not convex. In this case, one can always find $v' \in (\tilde{v}, \overline{v})$ and $v \in (v', \overline{v}]$ such that $H_{v',(F(v')^{n-1})'}(v) < 0$, because if v' belongs to a concave region of F^{n-1}, then $H_{v',(F(v')^{n-1})'}$ remains above F around v'. Let $v^\dagger$ denote the infimum among such v's. Clearly, $v^\dagger \in (\tilde{v}, \overline{v})$. In addition, since $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$

for all $v \in (\tilde{v}, \overline{v}]$, there must exist $\tilde{v}' \in (v^\dagger, \overline{v}]$ such that $W_{v^\dagger,(F(v^\dagger)^{n-1})'}(v) \geq 0$ for all $v \in [v^\dagger, \tilde{v}']$, and $W_{v^\dagger,(F(v^\dagger)^{n-1})'}(v^\dagger) = W_{v^\dagger,(F(v^\dagger)^{n-1})'}(\tilde{v}') = 0$. This implies that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is an MPC of F over $[v^\dagger, \overline{v}']$. ■

Figure 4 visualizes the arguments in Lemmas 3 and 4 (for the case in which $n = 2$). In the left panel, F (solid) is concave. Therefore, $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$ (dashed) stays uniformly above F, so it is not an MPC of F. In this case, we can find an MPC of F (dotted) by reducing the slope of $H^{n-1}_{\tilde{v},a}$ (that is, rotating down $H^{n-1}_{\tilde{v},a}$). In the right panel, $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$ stays uniformly below F. Then, simply increasing the slope of $H^{n-1}_{\tilde{v},a}$ does not work, as doing so yields $W_{\tilde{v},a}(\tilde{v}+\varepsilon) < 0$ for small ε, violating the MPC constraint. In this case, one can move $\tilde{v}$ to the right, which reduces the gap between F and $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ (observe how the dashed gray line shifts as $v^\dagger$ rises). If F^{n-1} is convex above $\tilde{v}$, then $W_{v^\dagger,(F(v^\dagger)^{n-1})'}$ remains bounded above zero for all values of $v^\dagger < \overline{v}$, in which case the only possible MPC that satisfies Lemmas 1 and 2 is F itself. Otherwise, by Lemma 4, there exist $v^\dagger \in (\tilde{v}, \overline{v})$ and $\tilde{v}' \in (v^\dagger, \overline{v}]$ such that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$.

Construction of G^*. Using $H_{\tilde{v},a}$ for different values of $\tilde{v}$ and a as building blocks, we recursively construct G^* in the *forward* direction. We begin by setting $\tilde{v} = \underline{v}$. Then, we construct G^* in the following three cases:

(1) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(\tilde{v}') \leq 0$ for some $v \in (\tilde{v}, \overline{v}]$: in this case, we set $G^*(v) = H_{\tilde{v},a^*}(v)$ for $v \in [\tilde{v}, \tilde{v}']$, where a^* and $\tilde{v}'$ are defined in Lemma 3. If there exist multiple solutions of $(a^*, \tilde{v}')$, we select the *smallest* a^*.

(2) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v})$, and F is not convex over $[\tilde{v}, \overline{v}]$: in this case, we set $G^*(v) = H_{v^\dagger,(F(v^\dagger)^{n-1})'}(v)$ for $v \in [\tilde{v}, \tilde{v}']$, where $v^\dagger$ and $\tilde{v}'$ are defined in Lemma 4. If there exist multiple solutions of $(v^\dagger, \tilde{v}')$, then we select the *smallest* $v^\dagger$.

(3) F is convex over $[\tilde{v}, \overline{v}]$: in this case, we set $G^* = F$ for $v \in [\tilde{v}, \overline{v}]$, which completes the construction.

In cases (1) and (2), the construction is complete if $G(\tilde{v}') = 1$. Otherwise, we construct G^* for the next *block* by setting $\tilde{v} = \tilde{v}'$ and repeating the same process.

This construction ensures that the resulting G^* satisfies Lemmas 1 and 2. For convexity of $(G^*)^{n-1}$, note that we choose the smallest a^* in case (1) and the smallest $v^\dagger$ in case (2). Therefore, the slope of G^* at the beginning of the next block is no less than the slope at the end of the previous block, producing the global convexity of $(G^*)^{n-1}$.

Uniqueness of G^*. Suppose that there exist two distribution functions, G_1 and G_2, that satisfy Lemmas 1 and 2. Let $\hat{v}$ denote the lowest point at which G_1 and G_2 diverge, that is, $\hat{v} \equiv \inf\{v :$

$G_1(v) \neq G_2(v)\}$. Without loss of generality, we assume that $G_1(v) < G_2(v)$ for v sufficiently close to $\hat{v}$. For notational simplicity, let

$$W_i(v) = \int_{\underline{v}}^{v} (F(x) - G_i(x))dx \text{ for } i = 1, 2.$$

Since $G_i \in \text{MPC}(F)$, $W_i(v) \geq 0$ for all v and $i = 1, 2$.

We make two observations about the G_i's. First, it must be that $G_i(\hat{v}) = F(\hat{v})$: otherwise, G_1^{n-1} and G_2^{n-1} must be linear with different slopes around $\hat{v}$, which violates the definition of $\hat{v}$. Then, obviously, $W_1(\hat{v}) = W_2(\hat{v}) = 0$. Second, G_1^{n-1} must be linear over $(\hat{v}, \hat{v} + \varepsilon)$ for some ε: otherwise, $G_1(v) = F(v)$ for $v \in (\hat{v}, \hat{v} + \varepsilon)$, in which case $G_2(v) > F(v)$, and therefore, $W_2(\hat{v} + \varepsilon) = \int_{\hat{v}}^{\hat{v}+\varepsilon}(F(x) - G_2(x))dx < 0$, violating the MPC constraint. Let $\hat{v}'$ denote the end point of the linear region (that is, $\hat{v}' \equiv \sup\{v : G_1^{n-1} \text{ is linear over } [\hat{v}, v]\}$).

Now observe that, since G_2^{n-1} is convex over its support, it must be that $G_2(v) > G_1(v)$ for all $v \in (\hat{v}, \hat{v}')$. We complete the proof by showing that this dominance leads to $W_2(v) < 0$ for some v. If G_1 does not end on $\hat{v}'$ (that is, $G_1(\hat{v}') < 1$), then G_1 must be an MPC of F over $[\hat{v}, \hat{v}']$. In this case, $W_1(\hat{v}') = \int_{\hat{v}}^{\hat{v}'}(F(x) - G_1(x))dx = 0$. However, then,

$$W_2(\hat{v}') = \int_{\hat{v}}^{\hat{v}'} (F(x) - G_2(x))dx = \int_{\hat{v}}^{\hat{v}'} (G_1 - G_2(x))dx < 0.$$

If G_1 ends on $\hat{v}'$ (that is, $G_1(\hat{v}') = 1$), then G_1 is an MPC of F over $[\hat{v}, \overline{v}]$. In this case, similarly to the previous case,

$$W_2(\overline{v}) = \int_{\hat{v}}^{\overline{v}} (F(x) - G_2(x))dx = \int_{\hat{v}}^{\overline{v}} (G_1(x) - G_2(x))dx < 0.$$

C Proof of Theorem 2

A tuple (p^*, G^*) is an equilibrium if and only if no compound deviation is profitable for an individual firm, that is,

$$\pi(p^*, G^*, p^*, G^*) \geq \pi(p_i, G_i, p^*, G^*) \text{ for all } p_i \in \mathcal{R}_+ \text{ and } G_i \in \text{MPC}(F).$$

Our proof strategy is—instead of directly considering all compound deviations—to find an optimal advertising strategy, denoted by $G_{p_i}^*$, that corresponds to each p_i and check only whether $\pi(p^*, G^*, p^*, G^*) \geq \pi(p_i, G_{p_i}^*, p^*, G^*)$ for all p_i. Clearly, this is necessary and sufficient for an equilibrium.

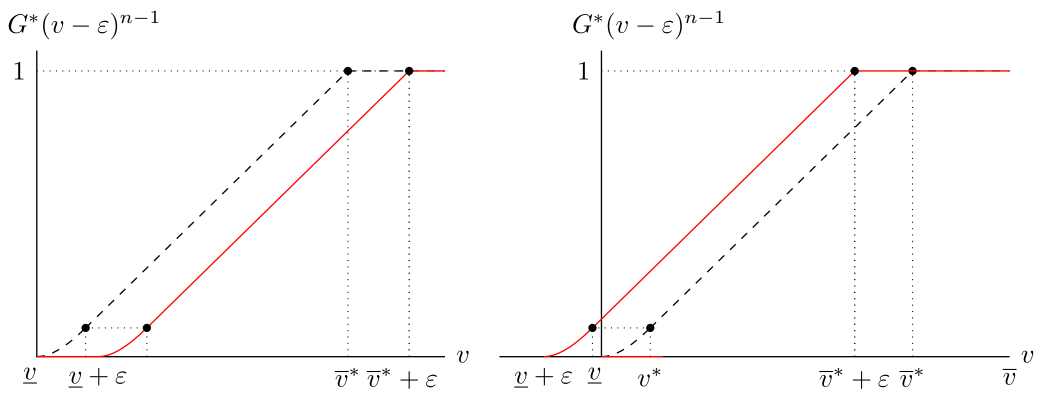


Figure 5: Both panels depict $G^*(v)^{n-1}$ (dashed) and $G^*(v-\varepsilon)^{n-1}$ (solid) when $\varepsilon \equiv p_i - p^*$. In the left panel, $p_i > p^*$ (so $\varepsilon > 0$), while $p_i < p^*$ (so $\varepsilon < 0$) in the right panel.

Step 1: Identifying an optimal advertising strategy to $p_i \neq p^*$. Let $G^*_{p_i}$ denote a solution to

$$\max_{G_i \in \text{MPC}(F)} D(p_i, G_i, p^*, G^*) = \int_{\underline{v}}^{\overline{v}} G^*(v - p_i + p^*)^{n-1} dG_i(v).$$

If f is log-concave, then F^{n-1} has quasi-concave density for any n. Therefore, by Corollary 2, G^* is characterized by $v^* \in [\underline{v}, \overline{v})$ such that $G^*(v) = F(v)$ if $v \leq v^*$ and $(G^*)^{n-1}$ is linear above v^*. This implies that $G^*(v - p_i + p^*)$ is convex over $[\underline{v}, \overline{v}^* + p_i - p^*]$, where $\overline{v}^*$ is the upper bound of $supp(G^*)$ (see Figure 5). Applying Theorem 3 to this structure, the following result is straightforward to obtain.

Lemma 5 *If $p_i - p^* \geq \overline{v} - \overline{v}^*$ then $G^*_{p_i} = F$. If $p_i - p^* \leq \mu_F - \overline{v}^*$ then $G^*_i = \delta_{\mu_F}$. Otherwise,*

$$G^*_{p_i}(v) = \begin{cases} F(v) & \text{if } v \leq \psi \\ F(\psi) & \text{if } v \in (\psi, \overline{v}^* + p_i - p^*) \\ 1 & \text{if } v \geq \overline{v}^* + p_i - p^*, \end{cases}$$

where ψ is the value such that $\mathbb{E}_F[v | v \geq \psi] = \overline{v}^* + p_i - p^*$.

In what follows, we let $\tilde{D}(p_i)$ denote firm i's demand when it plays $(p_i, G^*_{p_i})$ while all other firms play (p^*, G^*); that is, $\tilde{D}(p_i) \equiv D(p_i, G^*_{p_i}, p^*, G^*)$. As argued above, it suffices to show that

$$\pi(p^*, G^*, p^*, G^*) = p^* \tilde{D}(p^*) \geq \pi(p_i, G^*_{p_i}, p^*, G^*) = p_i \tilde{D}(p_i) \text{ for any } p_i.$$

Step 2: Defining a pseudo-demand function. Define the *pseudo-value* function $\varphi : \mathcal{R} \to \mathcal{R}_+$ as follows:

$$\varphi(v) = \begin{cases} G^*(v)^{n-1} & \text{if } v \leq \overline{v}^*, \\ 1 + (n-1)g^*(\overline{v}^*)(v - \overline{v}^*) & \text{if } v > \overline{v}^*. \end{cases}$$

In other words, φ follows $G^*(v)^{n-1}$ until $\overline{v}^*$ and then linearly extends it above $\overline{v}^*$. This function is well-defined because $g^*(\overline{v}^*) < \infty$. This function differs from $(G^*)^{n-1}$ in two ways. First, $\varphi(v)$ is not a distribution function, because $\varphi(v) > 1$ for $v > \overline{v}^*$. Second, and more importantly, whereas $(G^*)^{n-1}$ is convex only over $(-\infty, \overline{v}^*]$, φ is convex over $\mathcal{R}$.

Given φ, we define the *pseudo-demand* function $\widehat{D}(p_i)$ as follows:

$$\widehat{D}(p_i) \equiv \int \varphi(v - p_i + p^*)dF(v).$$

In other words, firm i's pseudo-demand is the measure of consumers that firm i would serve if it faced the pseudo-value function $\varphi(v - p_i + p^*)$, instead of $G^*(v - p_i + p^*)^{n-1}$, and used the fully informative advertising strategy F, instead of $G^*_{p_i}$. Clearly, the former property relatively raises the firm's demand, while the latter does the opposite. The following lemma shows that the overall effect is always non-negative and negligible if $p_i = p^*$.

Lemma 6 $\widehat{D}(p_i) \geq \tilde{D}(p_i)$ *for any* p_i*, and* $\widehat{D}(p^*) = \tilde{D}(p^*)$.

Proof. Since $\varphi(v) \geq G^*(v)^{n-1}$ for all v,

$$\int \varphi(v - p_i + p^*)dG^*_{p_i}(v) \geq \int G^*(v - p_i + p^*)^{n-1}dG^*_{p_i}(v) = \tilde{D}(p_i).$$

Meanwhile, since φ is globally convex and $G^*_{p_i} \in \text{MPC}(F)$,

$$\widehat{D}(p_i) = \int \varphi(v - p_i + p^*)dF(v) \geq \int \varphi(v - p_i + p^*)dG^*_{p_i}(v).$$

Combining these two inequalities leads to the first result.

For the second part, observe that, by construction, $\varphi(v)$ is linear whenever $G^*(v) \neq F(v)$ (i.e., for $v \in [v^*, \overline{v}]$) and so

$$\widehat{D}(p^*) = \int_{\underline{v}}^{\overline{v}} \varphi(v)dF(v) = \int_{\underline{v}}^{\overline{v}} \varphi(v)dG^*(v).$$

In addition, since φ coincides with $(G^*)^{n-1}$ over $supp(G^*) = [\underline{v}, \overline{v}^*]$, we have

$$\int_{\underline{v}}^{\overline{v}} \varphi(v) dG^*(v) = \int_{\underline{v}}^{\overline{v}^*} G^*(v)^{n-1} dG^*(v) = \tilde{D}(p^*),$$

which completes the proof. ∎

Step 3: Proving the optimality of p^*. Now we show that p^* maximizes $p_i \widehat{D}(p_i)$. This is sufficient for our purpose, because combining it with Lemma 6 yields

$$p^* \tilde{D}(p^*) = p^* \widehat{D}(p^*) \geq p_i \widehat{D}(p_i) \geq p_i \tilde{D}(p_i) \text{ for any } p_i.$$

The following result is crucial for the desired result.

Lemma 7 *If f is log-concave, then $\widehat{D}(p_i)$ is log-concave.*

Proof. If both $f(v)$ and $\varphi(v - p_i + p^*)$ are log-concave in v and p_i then, by Prékopa's theorem (1971), the pseudo-demand function $\widehat{D}(p_i)$–the convolution of φ and f–is also log-concave. Therefore, it suffices to show that $\varphi(v)$ is log-concave in v; note that it implies that $\varphi(v - p_i + p^*)$ is log-concave in both v and p_i. If $v < v^*$ then

$$\frac{\varphi'(v)}{\varphi(v)} = \frac{(n-1)F(v)^{n-2} f(v)}{F(v)^{n-1}} = (n-1)\frac{f(v)}{F(v)}.$$

Since F is log-concave (which is implied by log-concavity of f), φ'/φ is decreasing. If $v > v^*$, then $\phi(v)$ is linear and, therefore, clearly log-concave. The desired result then follows from the fact that φ is smooth, which ensures that φ'/φ is continuous around v^*. ∎

Given Lemma 7, $p_i \widehat{D}(p_i)$ is log-concave, because both $\log(p_i)$ and $\log(\widehat{D}(p_i))$ are concave. Then, it suffices to show that p^* satisfies the following first-order condition:

$$\frac{1}{p^*} + \frac{\widehat{D}'(p^*)}{\widehat{D}(p^*)} = 0 \Leftrightarrow \frac{1}{p^*} = -\frac{\widehat{D}'(p^*)}{\widehat{D}(p^*)}. \tag{11}$$

By Lemma 6, $\widehat{D}(p^*) = D(p^*, G^*, p^*, G^*) = 1/n$. The desired result is then implied by the following result, which states that a firm's ability to adjust its advertising strategy has no first-order effect on its demand around p^* (that is, a firm's optimal compound deviation is as profitable as its price-only deviation around p^*).

Lemma 8 *If f is log-concave, then*

$$\widehat{D}'(p^*) = \frac{\partial D(p_i, G^*, p^*, G^*)}{\partial p_i}|_{p_i=p^*}.$$

Proof. Consider $p_i = p^* + \varepsilon$. Then, the relevant demand functions are given by

$$\widehat{D}(p^* + \varepsilon) = \int \varphi(v - \varepsilon) dF(v) \text{ and } D(p^* + \varepsilon, G^*, p^*, G^*) = \int G^*(v - \varepsilon)^{n-1} dG^*(v).$$

Define $\Delta(\varepsilon) \equiv \widehat{D}(p^* + \varepsilon) - D(p^* + \varepsilon, G^*, p^*, G^*)$. Observe that $\Delta(0) = 0$ by Lemma 6. Then, it suffices to show that

$$\lim_{\varepsilon \to 0+} \Delta'(\varepsilon) = \lim_{\varepsilon \to 0-} \Delta'(\varepsilon) = 0.$$

Recall that if f is log-concave, then there exists $v^* \in [\underline{v}, \overline{v}]$ such that G^* coincides with F for $v < v^*$ and $(G^*)^{n-1}$ is linear for $v \geq v^*$. We analyze each of the following three cases: (1) $v^* = \overline{v}$, (2) $v^* = \underline{v}$, and (3) $v^* \in (\underline{v}, \overline{v})$.

Case 1: $v^* = \overline{v}$. In this case, F^{n-1} is globally convex, and thus,

$$G^*(v) = \begin{cases} F(v) & \text{if } v \leq \overline{v}, \\ 1 & \text{if } v > \overline{v}, \end{cases} \qquad \varphi(v) = \begin{cases} F(v)^{n-1} & \text{if } v \leq \overline{v}, \\ 1 + \hat{\beta}(v - \overline{v}) & \text{if } v > \overline{v}, \end{cases}$$

where $\hat{\beta} = (F(\overline{v})^{n-1})'$. The case with upward price deviations ($\varepsilon > 0$) is trivial: since $\varphi(v - \varepsilon) = G^*(v - \varepsilon)^{n-1}$ for all $v \in [\underline{v}, \overline{v}]$, $\Delta(\varepsilon) = 0$ for any $\varepsilon > 0$.

Now, consider the case in which $\varepsilon < 0$. In this case, the two demand functions differ only over $[\overline{v} + \varepsilon, \overline{v}]$. Formally,

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}}^{\overline{v}+\varepsilon} F(v - \varepsilon)^{n-1} dF(v) + \int_{\overline{v}+\varepsilon}^{\overline{v}} (1 + \hat{\beta}(v - \varepsilon - \overline{v})) dF(v),$$

$$D(p^* + \varepsilon, G^*, p^*, G^*) = \int_{\underline{v}}^{\overline{v}+\varepsilon} F(v - \varepsilon)^{n-1} dF(v) + \int_{\overline{v}+\varepsilon}^{\overline{v}} 1 dF(v).$$

Therefore,

$$\Delta(\varepsilon) = \int_{\overline{v}+\varepsilon}^{\overline{v}} \hat{\beta}(v - \epsilon - \overline{v}) dF(v),$$

and

$$\Delta'(\varepsilon) = \int_{\overline{v}+\varepsilon}^{\overline{v}} \hat{\beta} dF(v) = -\hat{\beta}(F(\overline{v}) - F(\overline{v} + \varepsilon)),$$

which goes to zero as $\varepsilon \to 0$.

Case 2: $v^* = \underline{v}$**.** In this case, Corollary 3 implies that $(G^*)^{n-1}$ is linear for its entire support. Specifically, G^* and φ are given by

$$G^*(v)^{n-1} = \begin{cases} 0 & \text{if } v \leq \underline{v}, \\ \hat{\gamma}(v - \underline{v}) & \text{if } v \in (\underline{v}, \overline{v}^*], \\ 1 & \text{if } v > \overline{v}^*, \end{cases} \qquad \varphi(v) = \begin{cases} 0 & \text{if } v \leq \underline{v}, \\ \hat{\gamma}(v - \underline{v}) & \text{if } v > \underline{v}, \end{cases}$$

where $\hat{\gamma} = 1/(\overline{v}^* - \underline{v})$ and $\overline{v}^*$ is determined by the same mean condition $(\mathbb{E}_F[x] = \mathbb{E}_{G^*}[v])$.

Consider an upward price deviation (i.e., $\varepsilon > 0$). Then, the two demand functions are

$$\begin{aligned} \widehat{D}(p^* + \varepsilon) &= \int_{\underline{v}+\varepsilon}^{\overline{v}} \hat{\gamma}(v - (\underline{v} + \varepsilon))dF(v), \\ D(p^* + \varepsilon, G^*, p^*, G^*) &= \int_{\underline{v}+\varepsilon}^{\overline{v}^*} \hat{\gamma}(v - (\underline{v} + \varepsilon))dG^*(v). \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta'(\varepsilon) &= -\int_{\underline{v}+\varepsilon}^{\overline{v}} \hat{\gamma} dF(v) + \int_{\underline{v}+\varepsilon}^{\overline{v}^*} \hat{\gamma} dG^*(v) \\ &= \hat{\gamma}\left(-(1 - F(\underline{v} + \varepsilon)) + (1 - G^*(\underline{v} + \varepsilon))\right), \end{aligned}$$

which converges to zero as $\varepsilon \to 0$.

Next, consider a downward price deviation $(\varepsilon < 0)$. In this case, the two demand functions are

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}}^{\overline{v}} \hat{\gamma}(v - (\underline{v} + \varepsilon))dF(v) = \frac{1}{2} - \hat{\gamma}\varepsilon,$$

and

$$\begin{aligned} D(p^* + \varepsilon, G^*, p^*, G^*) &= \int_{\underline{v}}^{\overline{v}^*+\varepsilon} \hat{\gamma}(v - (\underline{v} + \varepsilon))dG^*(v) + \int_{\overline{v}^*+\varepsilon}^{\overline{v}^*} 1 dG^*(v) \\ &= \int_{\underline{v}}^{\overline{v}^*+\varepsilon} \hat{\gamma}(v - (\underline{v} + \varepsilon))g^*(v)dv + 1 - G^*(\overline{v}^* + \varepsilon). \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta'(\varepsilon) = -\hat{\gamma} - \left(\hat{\gamma}(\overline{v}^* - \underline{v})g^*(\overline{v}^* + \varepsilon) - \int_{\underline{v}}^{\overline{v}^*+\varepsilon} \hat{\gamma} g^*(v)dv - g^*(\overline{v}^* + \varepsilon)\right) \\ = -\hat{\gamma} + \hat{\gamma}G^*(\overline{v}^* + \varepsilon), \end{aligned}$$

where the second equality is from the fact that $\hat{\gamma}(\overline{v}^* - \underline{v}) = 1$. The above formula converges to zero as $\varepsilon \to 0$, leading to the desired result.

Case 3: $v^* \in (\underline{v}, \overline{v})$. Recall from Corollary 2 that in this case, $(G^*)^{n-1}$ has a convex-linear structure. Formally, G^* and φ are given by

$$G^*(v)^{n-1} = \begin{cases} 0 & \text{if } v \le \underline{v}, \\ F(v)^{n-1} & \text{if } v \in (\underline{v}, v^*], \\ \alpha + \beta(v - v^*) & \text{if } v \in (v^*, \overline{v}^*], \\ 1 & \text{if } v > \overline{v}^*, \end{cases} \qquad \varphi(v) = \begin{cases} 0 & \text{if } v \le \underline{v}, \\ F(v)^{n-1} & \text{if } v \in (\underline{v}, v^*], \\ \alpha + \beta(v - v^*) & \text{if } v > v^*, \end{cases}$$

where $\alpha = F(v^*)^{n-1}$, $\beta = (F(v^*)^{n-1})'$, and $\overline{v}^*$ satisfies $\alpha + \beta(\overline{v}^* - v^*) = 1$.

Consider an upward price deviation (i.e., $\varepsilon > 0$). Then, the two demand functions are

$$\begin{aligned} \widehat{D}(p^* + \varepsilon) &= \int_{\underline{v}+\varepsilon}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} (\alpha + \beta(v - (v^* + \varepsilon)))dF(v), \\ D(p^* + \varepsilon, G^*, p^*, G^*) &= \int_{\underline{v}+\varepsilon}^{v^*} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dG^*(v) \\ &\quad + \int_{v^*+\varepsilon}^{\overline{v}^*} (\alpha + \beta(v - (v^* + \varepsilon)))dG^*(v). \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta(\varepsilon) = &\int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} (\alpha + \beta(v - (v^* + \varepsilon)))dF(v) \\ &- \int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dG^*(v) - \int_{v^*+\varepsilon}^{\overline{v}^*} (\alpha + \beta(v - (v^* + \varepsilon)))dG^*(v), \end{aligned}$$

and thus

$$\begin{aligned} \Delta'(\varepsilon) =& F(v^*)^{n-1} f(v^* + \varepsilon) + \int_{v^*}^{v^*+\varepsilon} (F(v-\varepsilon)^{n-1})' dF(v) - \alpha f(v^* + \varepsilon) - \int_{v^*+\varepsilon}^{\overline{v}} \beta dF(v) \\ &- \left(F(v^*)^{n-1} g^*(v^* + \varepsilon) + \int_{v^*}^{v^*+\varepsilon} (F(v-\varepsilon)^{n-1})' dG^*(v) - \alpha g^*(v^* + \varepsilon) - \int_{v^*+\varepsilon}^{\overline{v}^*} \beta dG^*(v) \right), \\ =& (F(v^*)^{n-1} - \alpha)(f(v^* + \varepsilon) - g^*(v^* + \varepsilon)) + \beta(F(v^* + \varepsilon) - G^*(v^* + \varepsilon)) \\ &+ \int_{v^*}^{v^*+\varepsilon} (F(v-\varepsilon)^{n-1})' dF(v) - \int_{v^*}^{v^*+\varepsilon} (F(v-\varepsilon)^{n-1})' dG^*(v), \end{aligned}$$

which converges to zero as $\varepsilon \to 0$.

It remains to analyze the case with downward price deviation ($\varepsilon < 0$). The two demand functions in this case are

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} (\alpha + \beta(v - (v^* + \varepsilon))) dF(v),$$

$$\begin{aligned} D(p^* + \varepsilon, G^*, p^*, G^*) = &\int_{\underline{v}}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*+\varepsilon}^{v^*} (\alpha + \beta(v - (v^* + \varepsilon))) dF(v) \\ &+ \int_{v^*}^{\overline{v}^*+\varepsilon} (\alpha + \beta(v - (v^* + \varepsilon))) dG^*(v) + \int_{\overline{v}^*+\varepsilon}^{\overline{v}^*} 1 dG^*(v). \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta(\varepsilon) = &\int_{v^*}^{\overline{v}} (\alpha + \beta(v - (v^* + \varepsilon))) dF(v) \\ &- \int_{v^*}^{\overline{v}^*+\varepsilon} (\alpha + \beta(v - (v^* + \varepsilon))) dG^*(v) - \int_{\overline{v}^*+\varepsilon}^{\overline{v}^*} 1 dG^*(v), \end{aligned}$$

and thus

$$\Delta'(\varepsilon) = -\int_{v^*}^{\overline{v}} \beta dF(v) - (\alpha + \beta(\overline{v} - v^*)) g^*(\overline{v}^* + \varepsilon) + \int_{v^*}^{\overline{v}} \beta dG^*(v) + g^*(\overline{v}^* + \varepsilon),$$

which equals zero since $F(v^*) = G^*(v^*)$ and $\alpha + \beta(\overline{v} - v^*) = 1$.

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Supplemental Material for Online Publication

D Quasiconvex Density

This appendix considers the case where F^{n-1} has quasiconvex density (i.e., F^{n-1} is initially concave and then convex). To avoid triviality, we assume that the density function $(F^{n-1})'$ is decreasing at least initially, so that it is either monotone decreasing or first decreasing and then increasing. See Figure 6 for the representative shape.

Equilibrium advertising. In the current case, an MPC of F can simultaneously satisfy Lemmas 1 and 2 only when it consists of an initial linear region and then the full information region. In other words, there exists $v^* \in (\underline{v}, \overline{v}]$ such that $(G^*)^{n-1}$ is linear over $[\underline{v}, v^*]$ and $G^*(v) = F(v)$ if $v \geq v^*$.

To identify the value of v^*, let $\underline{\beta}$ be the minimized value of $F(v)^{n-1}/(v - \underline{v})$ over $[\underline{v}, \overline{v}]$; that is, $\underline{\beta}$ is the minimal slope with which the linear function starting from $(\underline{v}, 0)$ crosses F^{n-1} again (see the densely dotted line in the right panel of Figure 6). In addition, let $\overline{\beta} \equiv \lim_{v \to \underline{v}+}(F(v)^{n-1})'$: note that $g^*(\underline{v}) \leq f(\underline{v})$ is necessary for $G^* \in \text{MPC}(F)$ (see the dash-dotted line in the right panel of Figure 6). For each $\beta \in (\underline{\beta}, \overline{\beta})$, let v_β be the point at which the linear function $\beta(v - \underline{v})$ crosses F^{n-1} from above. By the concave-convex shape of F, such a point, if exists, is unique. Then, define G_β as follows: if v_β exists then

$$G_\beta(v) \equiv \begin{cases} \big(\beta(v - \underline{v})\big)^{\frac{1}{n-1}} & \text{if } v \leq v_\beta, \\ F(v) & \text{if } v > v_\beta. \end{cases}$$

Otherwise,

$$G_\beta(v) \equiv \min\left\{ \big(\beta(v - \underline{v})\big)^{\frac{1}{n-1}}, 1 \right\}.$$

For the two boundary values, we let $G_{\underline{\beta}} \equiv \lim_{\beta \to \underline{\beta}+} G_\beta$ and $G_{\overline{\beta}} \equiv \lim_{\beta \to \overline{\beta}-} G_\beta$.

By construction, as β rises, G_β moves leftward and so continuously decreases in the sense of first-order stochastic dominance. It is also clear that $G_{\underline{\beta}}$ strictly dominates F, while $G_{\overline{\beta}}$ is strictly dominated by F. Therefore, there exists a unique value of $\beta^* \in [\underline{\beta}, \overline{\beta}]$ such that $\int v dG_{\beta^*}(v) = \mu_F$. Since G_{β^*} has the same mean as F and crosses F only once from below, $G_{\beta^*} \in \text{MPC}(F)$. It suffices to set $G^* = G_{\beta^*}$.

Equilibrium existence with $n = 2$ and symmetric quasiconvex density. Applying the above characterization to the case where $n = 2$ and F has symmetric quasiconvex density, it is immediate that $G^* = U[\underline{v}, \overline{v}]$. Exploiting the simple structure of G^*, we provide a necessary and sufficient

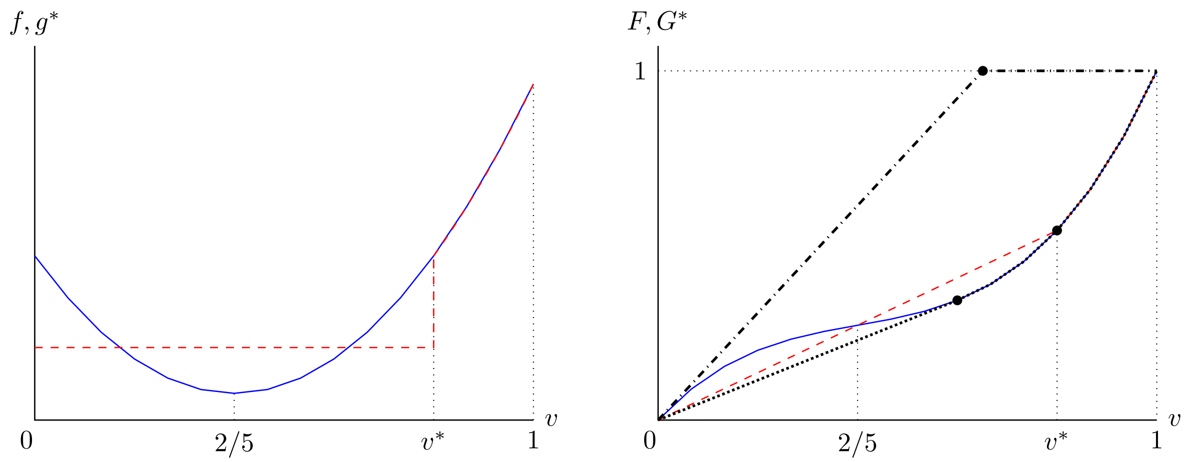


Figure 6: Equilibrium advertising strategy when F^{n-1} has quasiconvex density. In both panels, the blue solid curves represent the underlying distribution F, while the red dashed curves depict the equilibrium distribution G^*. In this figure, $n = 2$ and $f(v) = \frac{225}{28}\left(x - \frac{2}{5}\right)^2 + \frac{1}{4}$ over $[0, 1]$. In the right panel, the densely dotted curve represents $G_{\underline{\beta}}$, while the dash-dotted curve depicts $G_{\overline{\beta}}$.

condition under which (p^*, G^*) is indeed an equilibrium in our full model of advertising and pricing. Note that

$$\int g^*(v)^2 dv = \int_{\underline{v}}^{\overline{v}} \left(\frac{1}{\overline{v} - \underline{v}}\right)^2 dv = \frac{1}{\overline{v} - \underline{v}} \Rightarrow p^* = \frac{1}{2/(\overline{v} - \underline{v})} = \frac{\overline{v} - \underline{v}}{2} = \mu_F - \underline{v}.$$

Let $\pi(p_i)$ denote the maximal profit firm i can achieve by deviating to p_i and adjusting its advertising strategy accordingly; note that G^* may no longer be the firm's optimal advertising strategy. Formally,

$$\pi(p_i) \equiv \max_{G_i \in \text{MPC}(F)} p_i \int G^*(v - p_i + p^*) dG_i(v).$$

The following result shows that when $G^* = U[\underline{v}, \overline{v}]$, $\pi(p_i)$ takes a relatively simple form.

Lemma 9 *Let $\sigma = 1/(\overline{v} - \underline{v})$. If $n = 2$ and $G^* = U[\underline{v}, \overline{v}]$, then*

$$\pi(p_i) = \begin{cases} \sigma p_i (2p^* - p_i) & \text{if } p_i \in [0, p^*], \\ p_i \int \max\{0, \sigma(v - p_i + p^* - \underline{v})\} dF(v) & \text{if } p_i > p^*. \end{cases}$$

Proof. Suppose $p_i \in [0, p^*]$. Then, $G^*(v - p_i + p^*) \geq 0$ for any $v \geq \underline{v}$, and thus

$$G^*(v - p_i + p^*) = \min\{1, \sigma(v - p_i + p^* - \underline{v})\} \text{ for all } v \in [\underline{v}, \overline{v}].$$

Since this is concave over $[\underline{v}, \overline{v}]$, the degenerate distribution is *an* optimal strategy. Therefore,

$$\pi(p_i) = p_i G^*(\mu_F - p_i + p^*) = p_i \min\{1, \sigma(\mu_F - p_i + p^* - \underline{v})\} = \sigma p_i(2p^* - p_i),$$

where the last equality holds because

$$\sigma(\mu_F - p_i + p^* - \underline{v}) \leq \frac{\mu_F + p^* - \underline{v}}{\overline{v} - \underline{v}} = \frac{2(\mu_F - \underline{v})}{\overline{v} - \underline{v}} = 1 \text{ whenever } p_i \geq 0.$$

Now suppose $p_i > p^*$. In this case, $G^*(v - p_i + p^*) = \max\{0, \sigma(v - p_i + p^* - \underline{v})\}$ is convex over $[\underline{v}, \overline{v}]$. Therefore, F is an optimal advertising strategy. Combining this with the fact that $G^*(v - p_i + p^*) < 1$ for all $v \in [\underline{v}, \overline{v}]$, we arrive at

$$\pi(p_i) = p_i \int G^*(v - p_i + p^*) dF(v) = p_i \int \max\{0, \sigma(v - p_i + p^* - \underline{v})\} dF(v).$$

■

For equilibrium existence, it is necessary and sufficient that p^* maximizes $\pi(p_i)$. Clearly, $p_i(2p^* - p_i)$ is strictly increasing whenever $p_i < p^*$, so $\pi(p^*) \geq \pi(p_i)$ for any $p_i \leq p^*$ is guaranteed. This observation leads to the following result.

Lemma 10 *If $n = 2$ and $G^* = U[\underline{v}, \overline{v}]$ then (p^*, G^*) is an equilibrium if and only if the following function h, defined over $[0, \overline{v} - \underline{v}]$, is minimized at $\Delta = 0$:*

$$h(\Delta) \equiv (\mu_F - \underline{v} + \Delta) \int_{\underline{v}+\Delta}^{\overline{v}} (v - \underline{v} - \Delta) dF(v).$$

Proof. If $p_i \geq p^* + \overline{v} - \underline{v}$ then $\pi(p_i) = 0$, so it suffices to consider $p_i \in [p^*, p^* + \overline{v} - \underline{v}]$. Over the range, define $\Delta \equiv p_i - p^*$. Then, $\pi(p_i)$ can be rewritten as

$$\pi(p_i) = \pi(p^* + \Delta) = (\mu_F - \underline{v} + \Delta) \int_{\underline{v}+\Delta}^{\overline{v}} \sigma(v - \underline{v} - \Delta) dF(v) = \sigma h(\Delta).$$

Therefore, $\pi(p^*) \geq \pi(p_i)$ for all p_i if and only if $h(\Delta)$ is minimized at $\Delta = 0$. ■

Example 4 (Quadratic density) *Suppose $f(v) = a(v - 1/2)^2 + b$ over $[0, 1]$ for some $b \in [0, 1]$. For F to be a proper distribution,*

$$\int_0^1 f(v) dv = \int_0^1 \left[a \left(v - \frac{1}{2} \right)^2 + b \right] dv = \frac{a}{12} + b = 1 \Rightarrow a = 12(1 - b).$$

and, since $F(0) = 0$,

$$F(v) = \frac{a}{3}\left(v - \frac{1}{2}\right)^3 + bv + \frac{1-b}{2} = 4(1-b)v^3 - 6(1-b)v^2 + (3-2b)v.$$

Applying $\underline{v} = 0$, $\overline{v} = 1$, and $\mu_F = 1/2$ and arranging the terms, we arrive at

$$h(\Delta) = \left(\Delta + \frac{1}{2}\right)\int_{\Delta}^{1}(v - \Delta)dF(v) = \left(\Delta + \frac{1}{2}\right)\frac{(1-\Delta)^2}{2}(1 + 2(1-b)\Delta^2).$$

It can be shown that $h(0) = 1/4$, $h'(0) = 0$, and $h'(\Delta) < 0$ whenever $\Delta \in (0,1) = (0, \overline{v} - \underline{v})$. Therefore, $h(\Delta)$ is maximized at 0, so (p^*, G^*) is an equilibrium for any value of $b \in [0,1]$.

Example 5 (Equilibrium non-existence with more convex density) *Observe that*

$$h'(\Delta) = \int_{\underline{v}+\Delta}^{\overline{v}}(v - \mu_F - 2\Delta)dF(v) \text{ and } h''(\Delta) = (\mu_F - \underline{v} + \Delta)f(\underline{v} + \Delta) - 2(1 - F(\underline{v} + \Delta)).$$

Since $h'(0) = 0$, a necessary condition for $h(\Delta)$ to be maximized at 0 is $h''(0) \leq 0$. Consider $f(v) = a(v - 1/2)^{2m}$ for some natural number m. For F to be a proper distribution,

$$\int_0^1 f(v)dv = \int_0^1 a\left(v - \frac{1}{2}\right)^{2m} dv = \frac{a}{(2m+1)2^{2m}} = 1 \Rightarrow a = (2m+1)2^{2m}.$$

For this distribution, the aforementioned necessary condition holds if and only if

$$h''(0) = \mu_F f(0) - 2 = \frac{2m+1}{2} - 2 \leq 0 \Leftrightarrow m \leq \frac{3}{2}.$$

Together with Example 4, this implies that if $f(v) = a(v - 1/2)^{2m}$ over $[0,1]$ for some $m \in \mathcal{N}$ then (p^*, G^*) is an equilibrium if and only if $m = 1$ (i.e., f is a quadratic function).

E Symmetric Quadratic Density in the Perloff-Salop Model

Consider the Perloff-Salop model in which $n = 2$ and $f(v) = a(v - 1/2)^2 + b$ for some $b \in [0,1]$. As shown in Example 4 in Appendix D, F is a proper distribution if and only if $a = 12(1-b)$, and (p^*, G^*) is an equilibrium in our full model for any value of $b \in [0,1]$. In this appendix, we show that if b is sufficiently small then p^F is *not* an equilibrium, so a symmetric pure-price equilibrium fails to exist in the Perloff-Salop model.

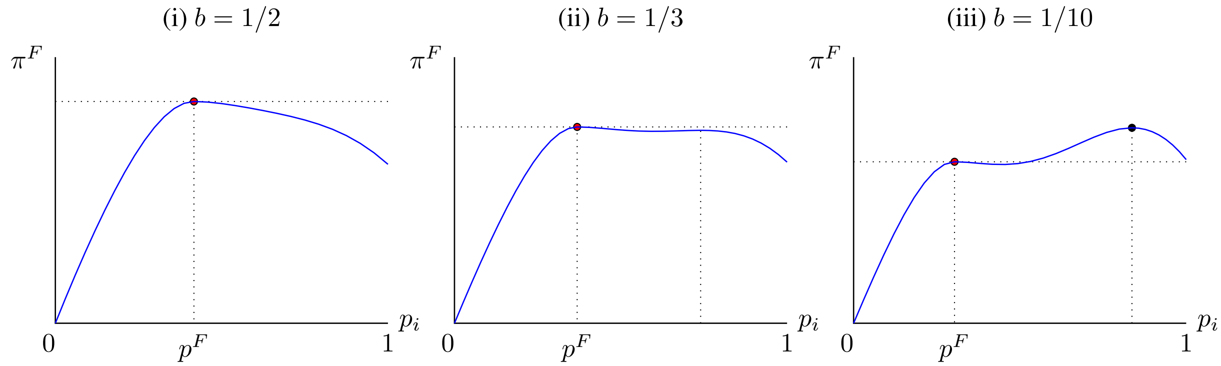


Figure 7: An individual firm's profit function $\pi^F(p_i)$ in the Perloff-Salop model when $n = 2$, $f(v) = a(x - 1/2)^2 + b$ for some $b \in (0, 1)$, and the other firm plays p^F.

By direct calculus, we have

$$p^F = \frac{1}{2\int f(v)^2 dv} = \frac{5}{2(9 - 8b + 4b^2)}.$$

A symmetric pure-price equilibrium exists if and only if p^F solves

$$\max_{p_i} \pi^F(p) \equiv p_i D(p_i, F, p^F, F) = p_i \int F(v - p_i + p^F) dF(v).$$

Since F is a cubic equation of v with a positive leading coefficient, π^F is effectively a quartic equation of p_i with a negative leading coefficient (i.e., opening downward), implying that $\pi^F(p_i)$ has one or two local maximums. It can be directly shown that $(\pi^F)'(p^F) = 0$ and $(\pi^F)''(p^F) < 0$, so p^F is necessarily a local maximizer.

There are the following three possibilities:

(i) p^F is the only local maximizer.

(ii) π_F has another local maximizer, which yields a lower profit than $\pi_F(p_F) = p_F/2$.

(iii) π_F has another local maximizer, which yields a higher profit than $\pi_F(p_F) = p_F/2$.

Clearly, p^F is an equilibrium with (i) or (ii), but not with (iii). As shown in Figure 7, all three cases arise, depending on the value of b. If b is relatively close to 1, then $\pi_F(p)$ is quasiconcave, so p^F maximizes $\pi_F(p)$ (see the left panel). If b is relatively small, then $\pi_F(p)$ has two peaks. Even in this case, p^F is the global maximizer if b is not so small (the middle panel). If b is sufficiently small, however, p^F does not maximize $\pi_F(p)$ any longer; that is, the other peak is higher than $\pi_F(p^F) = p^F/2$ (the right panel). According to our numerical analysis, this last case arises, so p^F is not an equilibrium, if and only if $b \leq 0.3043$.

F Truncated Exponential Distributions

This appendix provides a detailed welfare analysis for the case of truncated exponential distributions. Recall that $F(v) = \gamma(1 - e^{-\lambda v})$ over $[0, \overline{v}]$ for some $\lambda > 0$. Note that

$$F(\overline{v}) = \gamma(1 - e^{-\lambda \overline{v}}) = 1 \Leftrightarrow \gamma = \frac{1}{1 - e^{-\lambda \overline{v}}},$$

and

$$\mu_F = \int_0^{\overline{v}} v dF(v) = \overline{v} - \int_0^{\overline{v}} F(v) dv = \overline{v} - \gamma \overline{v} + \frac{\gamma(1 - e^{-\lambda \overline{v}})}{\lambda} = \frac{1}{\lambda} - (\gamma - 1)\overline{v} = \frac{1}{\lambda} - \frac{e^{-\lambda \overline{v}} \overline{v}}{1 - e^{-\lambda \overline{v}}}.$$

Clearly, μ_F increases and converges to $1/\lambda$ as $\overline{v}$ tends to ∞.

Price comparison. We show that for any $n \in [2, \overline{n})$, $\lambda > 0$, and $\overline{v} > 0$,

$$p^F > p^* \Leftrightarrow \int \left(F(v)^{n-1}\right)' dF(v) < \int \left(G^*(v)^{n-1}\right)' dG^*(v).$$

See the left panel of Figure 8 for the representative patterns of p^F and p^*.

If $n = 2$ then, by direct calculus,

$$\int_0^{\overline{v}} f(v) dF(v) = \int_0^{\overline{v}} (\gamma\lambda)^2 e^{-2\lambda v} dv = \frac{\gamma^2 \lambda}{2}(1 - e^{-2\lambda \overline{v}}) = \frac{\lambda(2\gamma - 1)}{2},$$

while

$$\int g^*(v) dG^*(v) = \int_0^{2\mu} \frac{1}{2\mu} dG^*(v) = \frac{1}{2\mu_F}.$$

It can be directly shown that $\int_0^{\overline{v}} f(v) dF(v) < \int g^*(v) dG^*(v)$ for any $\lambda > 0$ and $\overline{v} > 0$.

For $n > 2$, first observe that, since $f(v) = \lambda(\gamma - F(v))$,

$$\begin{aligned} & \int_0^{\overline{v}} \left(F(v)^{n-1}\right)' dF(v) = \int_0^{\overline{v}} f(v) dF(v)^{n-1} = \lambda \int_0^{\overline{v}} (\gamma - F(v)) dF(v)^{n-1} \\ = \ & \lambda \left(\gamma - \int_0^{\overline{v}} (n-1) F(v)^{n-1} dF(v)\right) = \lambda \left(\gamma - \frac{n-1}{n} \int_0^{\overline{v}} dF(v)^n \right) = \lambda \left(\gamma - \frac{n-1}{n}\right). \end{aligned}$$

Therefore,

$$p^F = \frac{1}{n \int_0^{\overline{v}} \left(F(v)^{n-1}\right)' dF(v)} = \frac{1}{\lambda(\gamma n - (n-1))}.$$

For p^*, notice that

$$\int_0^{\overline{v}^*} \left(G^*(v)^{n-1}\right)' dG^*(v) = \int_0^{v^*} \left(F(v)^{n-1}\right)' dF(v) + \int_{v^*}^{\overline{v}^*} \left(G^*(v)^{n-1}\right)' dG^*(v)$$

$$= \int_0^{\overline{v}} \left(F(v)^{n-1}\right)' dF(v) + \int_{v^*}^{\overline{v}^*} \left(G^*(v)^{n-1}\right)' dG^*(v) - \int_{v^*}^{\overline{v}} \left(F(v)^{n-1}\right)' dF(v).$$

Since $G^*(v)^{n-1}$ is linear over $[v^*, \overline{v}^*]$ with slope $\beta \equiv (F(v^*)^{n-1})'$ and $G^*(v^*) = F(v^*)$,

$$\int_{v^*}^{\overline{v}^*} \left(G^*(v)^{n-1}\right)' dG^*(v) = \int_{v^*}^{\overline{v}^*} \beta dG^*(v) = \beta(1 - F(v^*)).$$

For F, exploiting the fact that $f(v) = \lambda(\gamma - F(v))$ as before,

$$\begin{aligned}\int_{v^*}^{\overline{v}} \left(F(v)^{n-1}\right)' dF(v) &= \int_{v^*}^{\overline{v}} f(v) dF(v)^{n-1} = \lambda \int_{v^*}^{\overline{v}} (\gamma - F(v)) dF(v)^{n-1} \\ &= \lambda \left(\gamma(1 - F(v^*)^{n-1}) - \frac{n-1}{n}(1 - F(v^*)^n) \right).\end{aligned}$$

Now observe that

$$\begin{aligned}\int_{v^*}^{\overline{v}} v dF(x) &= \overline{v} - v^* F(v^*) - \int_{v^*}^{\overline{v}} F(v) dv = \overline{v} - v^* F(v^*) - \int_{v^*}^{\overline{v}} \gamma(1 - e^{-\lambda v}) dv \\ &= \overline{v} - v^* + v^*(1 - F(v^*)) - \gamma(\overline{v} - v^*) + \frac{\gamma}{\lambda}(e^{-\lambda v^*} - e^{-\lambda \overline{v}}) \\ &= v^*(1 - F(v^*)) - (\gamma - 1)(\overline{v} - v^*) + \frac{1 - F(v^*)}{\lambda},\end{aligned}$$

where the last equality is because $\gamma = 1/(1 - e^{-\lambda \overline{v}})$, so $1 - F(v^*) = \gamma(e^{-\lambda v^*} - e^{-\lambda \overline{v}})$. Meanwhile, since $G^*(v^*) = F(v^*)$, $G^*(\overline{v}^*) = 1$, and $(G^*)^{n-1} = F(v^*)^{n-1} + \beta(v - v^*)$ over $[v^*, \overline{v}^*]$,

$$\begin{aligned}\int_{v^*}^{\overline{v}^*} v dG^*(v) &= \overline{v}^* - v^* G^*(v^*) - \int_{v^*}^{\overline{v}^*} G^*(v) dv \\ &= v^*(1 - F(v^*)) + \frac{1 - F(v^*)^{n-1}}{\beta} - \int_{v^*}^{\overline{v}^*} \left[F(v^*)^{n-1} + \beta(v - v^*)\right]^{\frac{1}{n-1}} dv \\ &= v^*(1 - F(v^*)) + \frac{1 - F(v^*)^{n-1}}{\beta} - \frac{n-1}{n\beta}(1 - F(v^*)^n).\end{aligned}$$

Combining this with $\int_{v^*}^{\overline{v}} v dF(v) = \int_{v^*}^{\overline{v}^*} v dG^*(v)$ leads to

$$\beta(1 - F(v^*)) = \lambda \left(\beta(\gamma - 1)(\overline{v} - v^*) + 1 - F(v^*)^{n-1} - \frac{n-1}{n}(1 - F(v^*)^n) \right).$$

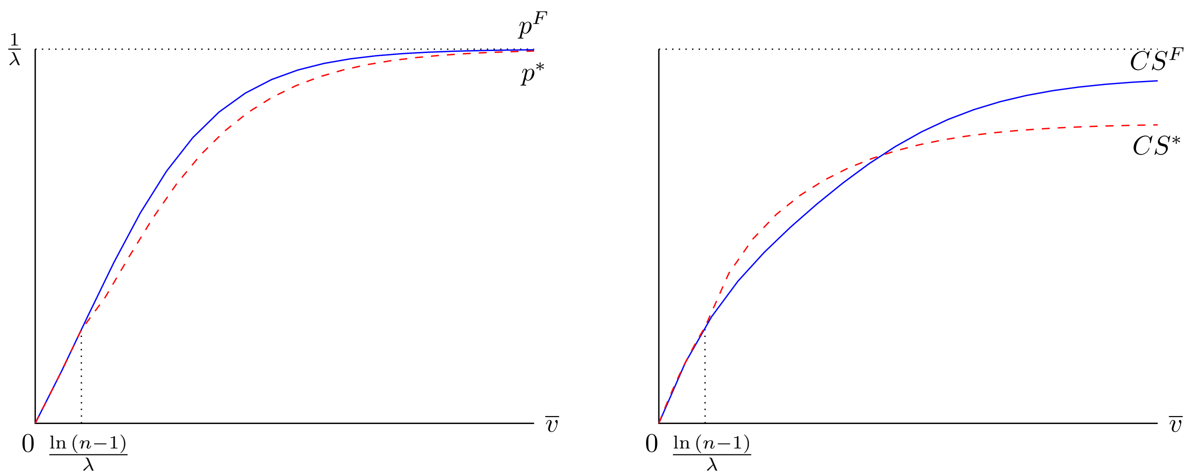


Figure 8: Price and welfare comparison for truncated exponential distributions. In both panels, $n = 3$ and $\lambda = 0.5$. Note that if $\overline{v} \leq \ln(n-1)/\lambda$ then F^{n-1} is convex over $[0, \overline{v}]$, in which case $G^* = F$.

It then follows that

$$\begin{aligned} & \frac{1}{\lambda}\left(\int_{v^*}^{\overline{v}^*} \left(G^*(v)^{n-1}\right)' dG^*(v) - \int_{v^*}^{\overline{v}} \left(F(v)^{n-1}\right)' dF(v)\right) \\ = \; & (\gamma - 1)\left(\beta(\overline{v} - v^*) - (1 - F(v^*)^{n-1})\right) > (\gamma - 1)\left(\beta(\overline{v}^* - v^*) - (1 - F(v^*)^{n-1})\right) = 0, \end{aligned}$$

where the inequality is because $\overline{v}^* < \overline{v}$ and the last equality is because $G(\overline{v}^*)^{n-1} = 1 = F(v^*)^{n-1} + \beta(\overline{v}^* - v^*)$. From this last observation, we conclude that

$$p^* = \frac{1}{\lambda\left(\gamma n - (n-1) + n(\gamma - 1)\left(\beta(\overline{v} - v^*) - (1 - F(v^*)^{n-1})\right)\right)} < p^F = \frac{1}{\lambda(\gamma n - (n-1))}.$$

From these solutions, it is also straightforward that as $\overline{v}$ tends to infinity, both p^* and p^F converge to $1/\lambda$, as depicted in the left panel of Figure 8.

Welfare comparison. To compute B^F and B^S, we make use of the following result.

Lemma 11 *If $F(x) = \gamma(1 - e^{-\lambda x})$ for $x \in [0, \overline{v}]$ then for any $v \in [0, \overline{v}]$,*

$$B(v) \equiv \int_0^v x dF^n(x) = vF(v)^n - \gamma^n v + \sum_{k=1}^{n} \frac{\gamma^{n-k} F(v)^k}{\lambda k}.$$

Proof. Integrating $B(v)$ by parts,

$$B(v) = vF(v)^n - \int_0^v F^n(x) dx.$$

The result then follows from the fact that, since $F(x) = \gamma - f(x)/\lambda$,

$$\begin{aligned}
\int_0^v F^n(x)dx &= \int_0^v F^{n-1}(x)\left(\gamma - \frac{f(x)}{\lambda}\right)dx = \gamma\int_0^v F^{n-1}(x)dx - \frac{1}{\lambda n}\int_0^v dF(x)^n = \\
&= \gamma\int_0^v F(x)^{n-2}\left(\gamma - \frac{f(x)}{\lambda}\right)dx - \frac{F(v)^n}{\lambda n} \\
&= \gamma^2\int_0^v F(x)^{n-2}dx - \frac{\gamma}{\lambda(n-1)}\int_0^v dF(v)^{n-1} - \frac{F(v)^n}{\lambda n} \\
&= \gamma^2\int_0^v F(x)^{n-3}\left(\gamma - \frac{f(x)}{\lambda}\right)dx - \frac{\gamma F(v)^{n-1}}{\lambda(n-1)} - \frac{F(v)^n}{\lambda n} \\
&= \ldots \\
&= \gamma^n\int_0^v dx - \frac{\gamma^{n-1}F(v)}{\lambda} - \ldots - \frac{F(v)^n}{\lambda n} = \gamma^n v - \sum_{k=1}^{n}\frac{\gamma^{n-k}F(v)^k}{\lambda k}.
\end{aligned}$$

■

Since $B^F = B(\overline{v})$, by Lemma 11,

$$B^F = -(\gamma^n - 1)\overline{v} + \sum_{k=1}^{n}\frac{\gamma^{n-k}}{\lambda k}.$$

B^* can be decomposed as follows:

$$B^* = \int_0^{v^*} v dG^*(v)^n + \int_{v^*}^{\overline{v}^*} v dG^*(v)^n = B(v^*) + \int_{v^*}^{\overline{v}^*} v dG^*(v)^n.$$

The first term is immediate from Lemma 11, while the latter term can be computed as follows:

$$\begin{aligned}
\int_{v^*}^{\overline{v}^*} v dG^*(v)^n &= \overline{v}^* - v^* F(v^*)^n - \int_{v^*}^{\overline{v}^*} G^*(v)^n dv \\
&= v^* + \frac{1 - F(v^*)^{n-1}}{\beta} - v^* F(v^*)^n - \int_{v^*}^{\overline{v}^*}\left[F(v^*)^{n-1} + \beta(v - v^*)\right]^{\frac{n}{n-1}} dv \\
&= v^*(1 - F(v^*)^n) + \frac{1 - F(v^*)^{n-1}}{\beta} - \frac{n-1}{(2n-1)\beta}(1 - F(v^*)^{2n-1}).
\end{aligned}$$

As shown in the right panel of Figure 8, typically, $CS^* > CS^F$ if $\overline{v}$ is relatively close to $\ln(n-1)/\lambda$, but $CS^* < CS^F$ for $\overline{v}$ sufficiently large. Intuitively, if $\overline{v}$ is sufficiently large then $p^F - p^*$ is sufficiently small (because both prices converge to $1/\lambda$), while $B^F - B^*$ persists (because G^* does not converge to F). Therefore, $CS^F > CS^*$ necessarily holds. If $\overline{v}$ is relatively small, however, $p^F - p^*$ can be significant and even larger than $B^F - B^*$.

G Equilibrium Convergence with Unbounded Support

This appendix considers the case where $supp(F)$ is unbounded above (i.e., $\overline{v} = \infty$) and shows that even in this case, the equilibrium distribution G^* of the advertising game converges to F as n tends to infinity, provided that F satisfies the following mild regularity condition.

Assumption 1 *There exists v' above which f is log-concave.*

Note that Assumption 1 regulates only the tail behavior of F; it is possible, for example, that f has multiple peaks before v'. In addition, whereas Assumption 1 ensures that F has a finite mean, most common distributions that violate Assumption 1 do not have a finite mean (see Bagnoli and Bergstrom, 2005). Finally, this is sufficient, but not necessary, for the desired convergence result.

A crucial implication of Assumption 1 is that there exists $\overline{n}$ such that if $n \geq \overline{n}$ then F^{n-1} has quasi-concave density: if n is sufficiently large then, for the same reason as in Corollary 1, F^{n-1} is convex over $[\underline{v}, v']$, that is, F^{n-1} has increasing density until v'. Meanwhile, Assumption 1 guarantees that F^{n-1} always has log-concave density above v'.

Let G_n^* denote the equilibrium advertising strategy when there are n firms. If $n \geq \overline{n}$ then, by Corollary 2, G_n^* has a simple cutoff structure: there exists v_n^* such that $G_n^*(v) = F(v)$ if $v \leq v_n^*$ and $(G_n^*)^{n-1}$ is linear above v_n^*. Let $\overline{v}_n^*$ denote the upper bound of $supp(G_n^*)$. Then, we have

$$G_n^*(v)^{n-1} = F(v_n^*)^{n-1} + \left(F(v_n^*)^{n-1}\right)' (v - v_n^*) \text{ for } v \in [v_n^*, \overline{v}_n^*],$$

which leads to

$$\begin{aligned}
\int_{v_n^*}^{\overline{v}_n^*} v dG_n^*(v) &= \overline{v}_n^* - v_n^* G_n^*(v_n^*) - \int_{v_n^*}^{\overline{v}_n^*} \left(F(v_n^*)^{n-1} + \left(F(v_n^*)^{n-1}\right)' (v - v_n^*)\right)^{\frac{1}{n-1}} dv \\
&= v_n^* + \frac{1 - F(v_n^*)^{n-1}}{(n-1)F(v_n^*)^{n-2} f(v_n^*)} - v_n^* F^*(v_n^*) - \frac{1 - F(v_n^*)^n}{nF(v_n^*)^{n-2} f(v_n^*)} \\
&= \frac{1 - nF(v_n^*)^{n-1} + (n-1)F(v_n^*)^n}{n(n-1)F(v_n^*)^{n-2} f(v_n^*)} + v_n^*(1 - F(v_n^*)).
\end{aligned}$$

Given the above cutoff structure of G_n^*, the same proof as for Corollary 2 applies, so $v_n^* \leq v_{n+1}^*$ for any $n \geq \overline{n}$. This means that for the desired convergence of G_n^* to F, it suffices to show that v_n^* grows unboundedly as n tends to infinity. Toward a contradiction, assume that there exists $\widehat{v}(< \infty)$ such that for any n, $v_n^* \leq \widehat{v}$. Then, as n tends to infinity, all $nF(v_n^*)^{n-1}$, $(n-1)F(v_n^*)^n$, and $n(n-1)F(v_n^*)^{n-2}$ converge to 0, so $\int_{v_n^*}^{\overline{v}_n^*} v dG_n^*(v)$ becomes arbitrarily large. But, this violates the MPC constraint that requires $\int_{v_n^*}^{\overline{v}_n^*} v dG_n^*(v) = \int_{v_n^*}^{\infty} v dF(v) < \infty$.

H Consumer Outside Option

This appendix provides two results about the case with consumer outside option, one on the equilibrium distribution G^* in the advertising game and the other on the equilibrium price p^*.

Equilibrium advertising. The following proposition generalizes Theorem 1 into the case where the consumer has a binding outside option.

Proposition 9 *Suppose that the consumer has the option of not purchasing any product. The advertising game in which the symmetric price is given as $p^* \in (\underline{v}, \overline{v})$ has an essentially unique symmetric equilibrium in which the equilibrium distribution G^* satisfies the following properties: for some $\underline{v} < v^\dagger < p^* < v^{\dagger\dagger} < \overline{v}$ and $\beta > 0$,*

(i) if $v \leq v^{\dagger\dagger}$ then

$$G^*(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \leq v^\dagger, \\ F(v^\dagger)^{n-1}, & \text{if } v \in (v^\dagger, p^*], \\ F(v^\dagger)^{n-1} + \beta(v - p^*), & \text{if } v \in (p^*, v^{\dagger\dagger}], \end{cases}$$

where

$$F(v^\dagger)^{n-1} + \beta(v^\dagger - p^*) = 0 \Leftrightarrow \beta = \frac{F(v^\dagger)^{n-1}}{p^* - v^\dagger}, \tag{12}$$

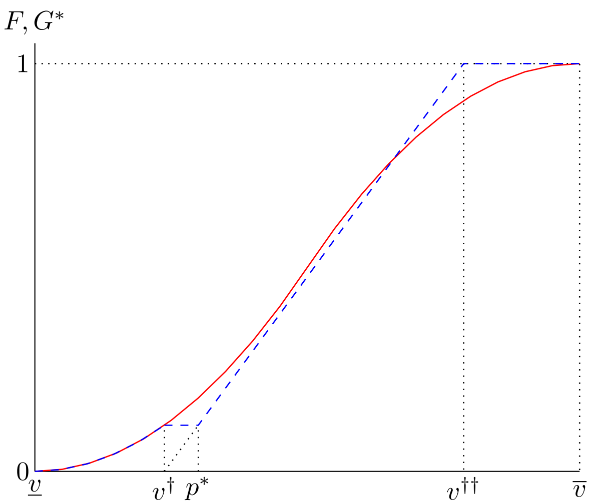
(ii) if $G^*(v^{\dagger\dagger}) = 1$, then G^* is an MPC of F over $[v^\dagger, \overline{v}]$, and

(iii) if $G^*(v^{\dagger\dagger}) < 1$, then $G^*(v^{\dagger\dagger}) = F(v^{\dagger\dagger})$, G^* is an MPC of F over $[v^\dagger, v^{\dagger\dagger}]$, and G^* satisfies Lemmas 1 and 2 for $v \geq v^{\dagger\dagger}$.

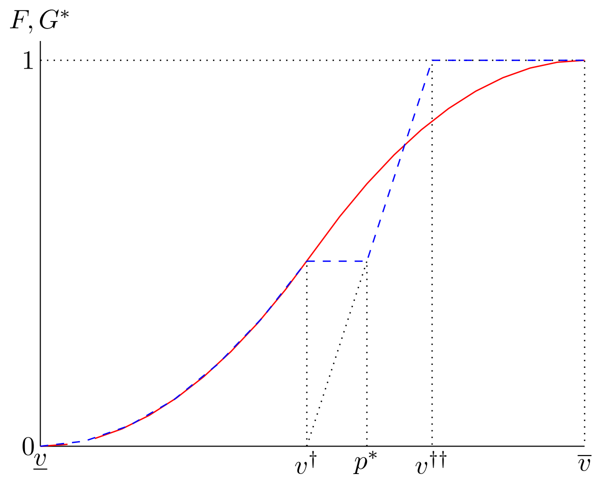
Furthermore, all symmetric equilibria of the game are identical to the above one for $v \geq v^\dagger$, with the equilibrium strategy being any MPC of F over $[\underline{v}, v^\dagger]$.

Figure 9 depicts G^* for the case with $n = 2$ and quasi-concave density. There are two noticeable changes from the baseline model in Section 4. First, G^* is flat (i.e., places no probability) on $[v^\dagger, p^*]$. This is natural because a firm has an incentive to give at least p^* to the consumer. Second, as clearly shown in Figure 9, $(G^*)^{n-1}$ is not convex over its support. This is because it suffices that $(G^*)^{n-1}$ is convex only above p^*.

One crucial property of G^* is that $(G^*)^{n-1}$ has the same (linear) slope at p^* as the line that connects $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$. In other words, if $(G^*)^{n-1}$ is linearly extended below p^*, then it must meet $(v^\dagger, 0)$ (see the black dotted line in each panel). To understand why this is necessary, suppose $(G^*)^{n-1}$ is flatter than the line between $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$. In this case, as when



(a) When $v^\dagger < v^*$



(b) When $v^\dagger > v^*$

Figure 9: An illustration of Proposition 9 for different values of p. In each panel, the red solid curve represents F, while the blue dashed curve depicts G^*.

$(G^*)^{n-1}$ is concave, it is profitable for a firm to concentrate local mass. On the contrary, if $(G^*)^{n-1}$ is steeper than the line between $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$, then it is profitable for a firm to reveal more product information. The collinearity is necessary and sufficient for neither to be profitable.

Proof of Proposition 9. Similarly to our equilibrium characterization in the main model (Appendix B), we first verify the optimality of G^* and then show the existence and (essential) uniqueness of the equilibrium.

Equilibrium verification. First, we show that G^* is a best response to itself and, therefore, an equilibrium in the advertising game.

Given that all other firms play G^*, an individual firm faces the following problem:

$$\max_{G_i \in \text{MPC}(F)} \int_{\underline{v}}^{\overline{v}} 1_{\{v \geq p\}} G^*(v)^{n-1} dG_i(v).$$

We utilize Theorem 3 to show the optimality of G^*. Let $u(v) = 1_{\{v \geq p\}} G^*(v)^{n-1}$. In addition, define $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}$ as follows:

$$\phi(v) = \begin{cases} 0 & \text{if } v \in [\underline{v}, v^\dagger), \\ F(v^\dagger)^{n-1} + \beta(v - p^*) & \text{if } v \in [v^\dagger, p^*), \\ u(v) & \text{if } v \in [p^*, \overline{v}^*], \\ \alpha(v - \overline{v}^*) + 1 & \text{if } v \in (\overline{v}^*, \overline{v}], \end{cases}$$

where $\overline{v}^*$ is the upper bound of support of G^*, and α and β are as defined in equation (9) and (12), respectively. By the structure of $(G^*)^{n-1}$, it is clear that ϕ is convex over $[\underline{v}, \overline{v}]$ and $\phi(v) \geq u(v)$ for all v. In addition, $\phi(v) = u(v)$ whenever $v \in supp(G^*) = [\underline{v}, v^\dagger] \cup [p^*, \overline{v}^*]$. Therefore, it suffices to show that

$$\int_{\underline{v}}^{\overline{v}} \phi(v) dG^*(v) = \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v).$$

Since $G^*(v) = F(v)$ for $v \in [\underline{v}, v^\dagger]$,

$$\begin{aligned}
&\int_{\underline{v}}^{\overline{v}} \phi(v) dG^*(v) - \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v) \\
&= \left(\int_{v^\dagger}^{v^{\dagger\dagger}} \phi(v) dG^*(v) - \int_{v^\dagger}^{v^{\dagger\dagger}} \phi(v) dF(v) \right) + \left(\int_{v^{\dagger\dagger}}^{\overline{v}} \phi(v) dG^*(v) - \int_{v^{\dagger\dagger}}^{\overline{v}} \phi(v) dF(v) \right).
\end{aligned}$$

We prove the desired result by showing that the right-hand side is zero. The terms inside the first set of parentheses are equal to zero because ϕ is linear and G^* is an MPC of F over $[v^\dagger, v^{\dagger\dagger}]$. Those in the second set of parentheses also sum to zero, because G^* satisfies the alternating structure of Lemma 2 for $v > v^{\dagger\dagger}$.

Equilibrium existence. Next, we prove the equilibrium existence by showing that there exists a unique pair of $v^\dagger \in (\underline{v}, p^*)$ and $\beta > 0$ such that the corresponding G^* satisfies the properties in the proposition.

We use a constructive method similar to the existence proof of the main model (Appendix B). For any $v' \in (\underline{v}, p^*)$ and $b \geq 0$, define a function $H(v; v', b)$ as follows:

$$H(v; v', b)^{n-1} = \begin{cases} F(v)^{n-1} & \text{if } v \in [\underline{v}, v'), \\ F(v')^{n-1} & \text{if } v \in [v', p^*), \\ \min\{F(v')^{n-1} + b(v - p^*), 1\} & \text{if } v \in [p^*, \overline{v}]. \end{cases}$$

Moreover, define

$$W(v; v', b) = \int_{\underline{v}}^{v} (F(v) - H(v; v', b)) dv.$$

Recall that $\mu_F(a) = \mathbb{E}_F[v | v \geq a]$, which is continuous in a and strictly increasing over $[\underline{v}, \overline{v}]$. The following lemma states a technical result that we utilize in this proof.

Lemma 12 $\lim_{b \to \infty} W(\overline{v}; v', b) < 0$ *if and only if* $\mu_F(v') > p^*$.

Proof.

$$\begin{aligned}\lim_{b\to\infty} W(\overline{v};v',b) &= \lim_{b\to\infty}\int_{v'}^{\overline{v}}(F(v)-H(v;v',b))dv \\ &= \int_{v'}^{\overline{v}}F(v)dv - \left(\int_{v'}^{p^*}F(v')dv + \int_{p^*}^{\overline{v}}1dv\right) \\ &= (\overline{v}-v'F(v')) - \int_{v'}^{\overline{v}}vdF(v) - (F(v')(p^*-v')+(\overline{v}-p^*)) \\ &= (1-F(v'))(p^*-\mu_F(v')).\end{aligned}$$

■

Define $\underline{v}^* = \min\{\nu \in [\underline{v}, p^*) : \mu_F(\nu) \geq p^*\}$. Observe that $\underline{v}^*$ always exists since $\mu_F(p^*) > p^*$. Using the next two lemmas, we show that there exist a unique $v^\dagger \in (\underline{v}^*, p^*)$ and $\beta > 0$ such that $H(v; v^\dagger, \beta) = G^*(v)$ for $v \in [\underline{v}, v^{\dagger\dagger}]$.

The first lemma states that for any $v' \in (\underline{v}^*, p^*)$, there exists $\tilde{b}(v')$ such that $H(v; v', \tilde{b}(v'))$ can be used for the construction of G^*.

Lemma 13 *For each $v' \in (\underline{v}^*, p^*)$, there exists a $\tilde{b}(v') \in (0, \infty)$ such that (a) $W(v; v', \tilde{b}(v')) \geq 0$ for all $v \in [\underline{v}, \overline{v}]$, and (b) $W(\hat{v}^*; v', \tilde{b}(v')) = 0$ for some $\hat{v}^* \in (p^*, \overline{v}]$, and (c) $H(v; v', \tilde{b}(v'))^{n-1}$ is concave for all $v \geq p^*$.*

Proof. Observe that $W(v; v', b) = 0$ for any $v \in [\underline{v}, v']$ and $W(v; v', b) > 0$ for any $v \in (v', p^*]$. Also, observe that for any $v \in (p^*, \overline{v}]$, $W(v; v', b)$ is continuous and strictly decreasing in b. Since $W(v; v', b = 0) > 0$ for any $v \in (p^*, \overline{v}]$ (since F is strictly increasing in v) and $\lim_{b\to\infty} W(\overline{v}; v', b) < 0$ (from Lemma 12 and the fact that $\mu_F(v') > p^*$ for any $v' > \underline{v}^*$), by the Intermediate Value Theorem, there exists $\tilde{b}(v') \in (0, \infty)$ that satisfies the conditions in the claim.
■

If there exist multiple values of $\tilde{b}(v')$, we pick the smallest value. Similar to equilibrium construction in the baseline model, the smallest such $\tilde{b}(v')$ is the only one that can be used to construct G^* which is concave for all $v \geq p^*$.

We now find the unique $v^\dagger$ such that $H(v; v^\dagger, \tilde{b}(v^\dagger)) = G^*(v)$ for $v \in [\underline{v}, v^{\dagger\dagger}]$. For each $v' \in (\underline{v}^*, p^*)$, define $\kappa(v')$ such that

$$F(v')^{n-1} + \tilde{b}(v')(\kappa(v') - p^*) = 0 \Longleftrightarrow \kappa(v') = p^* - \frac{F(v')^{n-1}}{\tilde{b}(v')}. \tag{13}$$

Observe that if $v^\dagger \in (\underline{v}^*, p^*)$ satisfies $\kappa(v^\dagger) = v^\dagger$, then $v^\dagger$ and $\beta = \tilde{b}(v^\dagger)$ satisfy (12).

Lemma 14 *(i)* $\kappa(v')$ *is continuous and decreasing in* v'*;*
(ii) $\lim_{v' \to \underline{v}^*} \kappa(v') = p^*$*;*
(iii) $\lim_{v' \to p^*} \kappa(v') < p^*$.

Proof. *(i)* Since $F(v')$ is continuous and increasing in v', from (13), it suffices to show that $\tilde{b}(v')$ is continuous and decreasing in v'. The continuity of $\tilde{b}(v')$ is naturally derived from the continuity of $W(v; v', b)$ in both v' and b. For monotonicity, suppose to the contrary that there exists $v', v'' \in (\underline{v}^*, p^*)$ $(v' < v'')$ such that $\tilde{b}(v') < \tilde{b}(v'')$. Let $\hat{\nu} \in (p^*, \overline{v}]$ be such that $W(\hat{\nu}; v', \tilde{b}(v')) = 0$. However, from the definition of $H(v; v', b)$, it must be that $H(v; v', \tilde{b}(v')) \leq H(v; v'', \tilde{b}(v''))$ for all $v \in [\underline{v}, \hat{\nu})$ with strict inequality holding at least for $v \in (v', p^*)$. Therefore, it must be that $W(\hat{\nu}; v'', \tilde{b}(v'')) < 0$, contradicting to the definition of $\tilde{b}(v'')$.

(ii) Suppose that $\mathbb{E}_F[v] \geq p^*$, which implies $\underline{v}^* = \underline{v}$. Then it must be that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') > 0$, as $W(v; \underline{v}, b = 0) > 0$ for any $v > (p^*, \overline{v}]$. Since $\lim_{v' \to \underline{v}} F(v')^{n-1} = 0$, from (13) we have the desired result.

Now, suppose that $\mathbb{E}_F[v] < p^*$. Then, it must be that $\underline{v}^* > \underline{v}$ and $\mu_F(\underline{v}^*) = p^*$. We claim that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') = \infty$. By Lemma 12, $\mu_F(\underline{v}^*) = p^*$ implies that $\lim_{b \to \infty} W(\overline{v}; \underline{v}^*, b) = 0$. Since $\lim_{b \to \infty} H(v; v^*, b) = 1$ for any $v \in (p^*, \overline{v}]$, it follows that for any $v \in (p^*, \overline{v})$, $\lim_{b \to \infty} W(v; \underline{v}^*, b) > 0$. Then, for any *finite* b and $v \in (p^*, \overline{v}]$, we have

$$\lim_{v' \to \underline{v}^*} W(v; v', b) = W(v; \underline{v}^*, b) > \lim_{b \to \infty} W(v; \underline{v}^*, b) > 0,$$

since $W(\cdot; v', b)$ is continuous in v' and strictly decreasing in b. Therefore, it must be that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') = \infty$, which implies that $\lim_{v' \to \underline{v}^*} \kappa(v') = p^*$.

(iii) Observe that $\lim_{v' \to p^*} \tilde{b}(v') > 0$, since $W(v; v', b = 0) > 0$ for any $v \in (p^*, \overline{v}]$. Therefore, from (13), $\lim_{v' \to p^*} \kappa(v') < p^*$. ■

Lemma 14 and the Intermediate Value Theorem together imply that there exists a unique $v^\dagger \in (\underline{v}, p^*)$ such that $\kappa(v^\dagger) = v^\dagger$.

We are ready to construct G^* given $v^\dagger$ and $\beta = \tilde{b}(v^\dagger)$. Let $v^{\dagger\dagger} = \max\{v > p^* : W(v; v^\dagger, \tilde{b}(v^\dagger)) = 0\}$. By Lemma 13, $v^{\dagger\dagger}$ always exists and $v^{\dagger\dagger} \leq \overline{v}$. For $v \in [\underline{v}, v^{\dagger\dagger}]$, construct G^* as

$$G^*(v)^{n-1} = \begin{cases} F(v)^{n-1} & \text{if } v < v^\dagger, \\ F(v^\dagger)^{n-1} & \text{if } v \in [v^\dagger, p^*), \\ \min\{F(v^\dagger)^{n-1} + \tilde{b}(v^\dagger)(v - p^*), 1\} & \text{if } v \in [p^*, v^{\dagger\dagger}]. \end{cases}$$

If $v^{\dagger\dagger} = \overline{v}$, then the construction of G^* is complete. If $v^{\dagger\dagger} < \overline{v}$, construct G^* for $v \in (v^{\dagger\dagger}, \overline{v}]$

using a method identical to one used in the existence proof of the main model (Appendix B). By the construction of $v^\dagger$ and $\tilde{b}(v^\dagger)$, it is straightforward to verify that G^* satisfies the properties in Proposition 9.

Essential uniqueness of equilibrium. Let G be a symmetric equilibrium strategy of the advertising game. We characterize the set of all equilibria of the game using the following steps:

Step 1. *G must not coincide with F at around p^*:* Suppose to the contrary that $G(v) = F(v)$ over $[p^* - \varepsilon, p^* + \varepsilon]$. Then it is clearly profitable for each firm to move the probability mass around p^* and assigns it to p^* in the way of mean-preserving contraction.

Let

$$M = \{(v', v'')|G(v) = F(v) \text{ for } v = v', v'', \text{and } G \text{ is an MPC of } F \text{ over } [v', v'']\}.$$

Then Step 1 implies that there must exist $(v_\alpha, v_\beta) \in M$ such that $v_\alpha < p^* < v_\beta$ and $(v'_\alpha, v'_\beta) \notin M$ for any $v'_\alpha \geq v_\alpha, v'_\beta \leq v_\beta$, and $(v'_\alpha, v'_\beta) \neq (v_\alpha, v_\beta)$.

Step 2. *G^{n-1} must be strictly increasing and linear on $[p^*, v_\beta] \cup supp(G)$:* It is a straightforward application of Lemma 2.

Step 3. *$G(v_\alpha) = G(p^*)$:* Suppose to the contrary that $G(v_\alpha) < G(p^*)$. Since G must not have an atom at p^*, it must be that G has a probability mass on (v_α, p^*). However, then each firm finds it profitable to offer a mean-preserving contraction of G, which takes the probability mass on $(v_\alpha, p^*) \cup (p^*, p^* + \varepsilon)$ for a sufficiently small $\varepsilon > 0$ (Step 2 implies that G has probability mass on $(p^*, p^* + \varepsilon)$) and put it on p^*.

Define $\hat{\kappa}$ be the horizontal intercept of the line extended from the linear part of G^{n-1} over $[p^*, v_\beta]$, that is,

$$G(v_\alpha)^{n-1} + \hat{b}(\hat{\kappa} - p^*) = 0 \Longleftrightarrow \hat{\kappa} = p^* - \frac{G(v_\alpha)^{n-1}}{\hat{b}},$$

where $\hat{b}$ is the slope of the G^{n-1} over $[p^*, v_\beta]$.

Step 4. *$\hat{\kappa} = v_\alpha$:* Suppose to the contrary that $\hat{\kappa} \neq v_\alpha$. If $\hat{\kappa} > v_\alpha$, then each firm has an incentive to take the probability mass on $(p^*, p^* + \varepsilon)$ and take its mean-preserving spread over $(v_\alpha, v_\alpha + \varepsilon) \cup (v_\beta - \varepsilon, v_\beta)$, since doing so yields a higher expected demand. Conversely, if $\hat{\kappa} < v_\alpha$, each firm has an incentive to take the probability mass on $(v_\alpha - \varepsilon, v_\alpha) \cup (v_\beta - \varepsilon, v_\beta)$ and put it on p^* in the sense of mean-preserving contraction, inducing a greater expected payoff. ■

Equilibrium price. Unlike in the baseline model, the equilibrium distribution G^* in Proposition 9 depends on the given symmetric price p^*. This implies that the equilibrium price p^* is a fixed point such that G^* is an equilibrium distribution given p^* and also produces p^* in the Perloff-Salop model. The following result shows that in the representative case where F^{n-1} has quasi-concave density, such a fixed point exists.

Proposition 10 *Suppose the consumer has a binding outside option: the equilibrium price in the baseline model, denoted by p_0^*, lies between $\underline{v}$ and $\overline{v}$. If F^{n-1} has strict quasi-concave density, then there exists $p^* \in [\underline{v}, \overline{v}]$ and $G^* \in MPC(F)$ such that (a) G^* is a symmetric equilibrium in the advertising game given p^* and (b) p^* satisfies the equilibrium first-order pricing formula in the Perloff-Salop model given G^*.*

Proof. Let $G^*(v;p)$ be the equilibrium distribution in the advertising game given p. Define a self-map $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, so that $\Phi(p)$ satisfies the equilibrium first-order pricing formula given $G^*(\cdot;p)$; if p is such that $v^\dagger > \underline{v}$ then

$$\Phi(p) \equiv \frac{\frac{1}{n}\left(1 - G^*(v^\dagger)^n\right)}{\int 1_{\{v \geq p\}} \left(G^*(v;p)^{n-1}\right)' dG^*(v;p)}.$$

Clearly, $\Phi(p) = p_0^* \geq \underline{v}$ for any $p \leq \underline{v}$. In addition, $\Phi(p) \leq \overline{v}$ for any $p \leq \overline{v}$: this is simply because a firm obtains zero profit if it charges a price above $\overline{v}$ when the consumer can refuse to buy any product. Given this, it suffices to show that Φ is continuous in p; if so, Brouwer's fixed point theorem ensures the existence of a fixed point in $[\underline{v}, \overline{v}]$.

We first denote several important variables as functions of a symmetric price p. Let $H(v; v', p, b)$ and $W(v; v', p, b)$ be the functions $H(v; v', b)$ and $W(v; v', b)$ given a price p. Define $v^\dagger(p)$ and $v^{\dagger\dagger}(p)$ in a similar manner. Let $\tilde{b}(v';p)$ be the smallest value that satisfies conditions in Lemma 13. Also, let $\kappa(v';p)$ be the horizontal intercept of the extended line of the linear part of $H(v; v', p, \tilde{b}(v';p))^{n-1}$ as defined in (13), that is,

$$\kappa(v';p) = p - \frac{F(v')^{n-1}}{\tilde{b}(v';p)}.$$

For the desired continuity result, it suffices to prove that for all $v \in [\underline{v}, \overline{v}]$, $G^*(v;p)$ is continuous in p. In the following, we prove this by showing two results. First, $v^\dagger(p)$ and $v^{\dagger\dagger}(p)$ are continuous for all but finite points of p. From the construction of $G^*(v;p)$ in Proposition 9, continuity of $v^\dagger(p)$ and $v^{\dagger\dagger}(p)$ guarantees the continuity of $G^*(v;p)$ in p. Second, for the points at which either of $v^\dagger(p)$ or $v^{\dagger\dagger}(p)$ is not continuous, we directly show that $G^*(v;p)$ is continuous in p.

First, we claim that $v^\dagger(p)$ is continuous for any p. Given Lemma 14, it suffices to show that $\kappa(v';p)$ is continuous in p. This follows from the continuity of $\tilde{b}(v';p)$ in p, which in turn follows from the fact that for every $v \in [\underline{v}, \overline{v}]$, $W(v; v', p, b)$ is continuous in p.

Next, we show that $v^{\dagger\dagger}(p)$ is continuous for all but finite points of p, and even when $v^{\dagger\dagger}(p)$ is discontinuous, the value of $G^*(v;p)$ for any v is continuous in p. Since F^{n-1} has a strict quasi-concave density, for any values of v', p and b, $H(v; v', p, b)$ crosses F^{n-1} from above at most once. Let us analyze the continuity of $v^{\dagger\dagger}(p)$ in the following two cases:

1. If $H(v; v^\dagger(p), p, \tilde{b}(v^\dagger(p); p)$ does not cross F^{n-1} from above, then $W(v; v^\dagger(p), p, \tilde{b}(v^\dagger(p); p))$ does not have an interior local minimum, and thus $v^{\dagger\dagger}(p) = \overline{v}^*$. Then it naturally follows from the continuity of $v^\dagger$ and $\tilde{b}(v^\dagger(p); p)$ that $v^{\dagger\dagger}(p)$ is continuous in p.

2. If $H(v; v^\dagger(p), p, \tilde{b}(v^\dagger(p); p)$ crosses F^{n-1} from above at $\hat{p} \in (p, \overline{v})$, then $W(v; v^\dagger(p), p, \tilde{b}(v^\dagger(p); p))$ has an interior local minimum at $\hat{v}$. In this case, either one of the following two subcases arises (we abuse notation by denoting $W(v; p) = W(v; v^\dagger(p), p, \tilde{b}(v^\dagger(p); p)))$:

$$\text{(a) } W(\hat{v}; p) > 0 \text{ and } W(\overline{v}; p) = 0 \implies v^{\dagger\dagger} = \overline{v}^*;$$
$$\text{(b) } W(\hat{v}; p) = 0 \text{ and } W(\overline{v}; p) \geq 0 \implies v^{\dagger\dagger} = \hat{v}.$$

   Therefore, if $v^{\dagger\dagger}$ can be discontinuous at some p', then the global minimum of $W(\overline{v}; p)$ must change from $\hat{v}$ to $\overline{v}$ (or vice versa) at around $p = p'$. Therefore, it must be that $W(\hat{v}; p') = W(\overline{v}; p') = 0$. Then the construction of $G^*(v'; p)$ in Proposition 9 implies that $G^*(v'; p)$ continuously changes from a piecewise linear function above p (with a kink at $\hat{v}$ to a linear function above p.

■

I Equilibrium Characterization in the Asymmetric Case

In this appendix, we present an equilibrium characterization result of the advertising game when $n = 2$ and the firms' value distributions are asymmetric. Suppose that the consumer's value for firm i's $(i = 1, 2)$ product is drawn according to the cumulative distribution function F_i. We assume that F_i has a compact and convex support $[\underline{v}_i, \overline{v}_i]$. Other regularity assumptions on the distribution made in the main model are similarly applied here. We also assume that F_2 first-order stochastically dominates F_1. Finally, we assume a non-trivial case in which $\mathbb{E}_{F_2}[v] < \overline{v}_1$.

Let G_i^* be firm i's equilibrium strategy in the advertising game. We first present extensions of the equilibrium necessary conditions of the main model, Lemmas 1 and 2, to the asymmetric two-firm case. Their proofs naturally follow from those of the main model, and thus are omitted.

Lemma 15 *For each $i = 1, 2$, G_i^* is convex over $\cap_i supp(G_i^*)$, which is itself a convex set.*

Lemma 16 *For almost all $v \in \cap_i supp(G_i^*)$, there exists $\varepsilon > 0$ such that either G_i^* coincides with F_i or G_i^* is linear over $(v - \varepsilon, v + \varepsilon)$. For each interval I of $\cap_i supp(G_i^*)$ on which G_i^* is linear, if*

I is in the interior of $\cap_i supp(G_i^*)$, G_j^* must be an MPC of F_j over I. If $\inf \cap_i supp(G_i^*) \in I$ (resp. $\sup \cap_i supp(G_i^*) \in I$), then there exists some $\hat{v}_i < \inf \cap_i supp(G_i^*)$ (resp. $\hat{v}_i > \sup \cap_i supp(G_i^*)$) such that G_j^* must be an MPC of F_j over $[\inf \cap_i supp(G_i^*), v_i] \cap I$ (resp. $I \cap [\sup \cap_i supp(G_i^*), v_i]$).

Contrary to the symmetric case where Lemmas 1 and 2 imply the existence of unique symmetric equilibrium, Lemmas 15 and 16 guarantees neither equilibrium existence or uniqueness. Further, characterizing equilibrium structure for general F_i is extremely difficult. Below, we present a partial equilibrium characterization result for the two special case, in which both F_1 and F_2 are convex or concave.

I.1 Concave F_i

First consider a case in which both F_1 and F_2 are concave.

Proposition 11 *Suppose that F_1 and F_2 are both concave. Then any equilibrium of the advertising game has the following structure: there exist v_a and v_b ($\underline{v}_2 < v_a < v_b < \overline{v}_1$) such that*

$$G_1^*(v) = \begin{cases} H(v) & \text{if } v < v_a, \\ F_1(v_1) + \frac{(1-F_1(v_1))(v-v_a)}{v_b-v_a} & \text{if } v \in [v_a, v_b), \\ 1 & \text{if } v \geq v_b, \end{cases} \qquad G_2^*(v) = \begin{cases} 0 & \text{if } v < v_a, \\ \frac{v-v_a}{v_b-v_a} & \text{if } v \in [v_a, v_b), \\ 1 & \text{if } v \geq v_b, \end{cases}$$

where H is an increasing and right-continuous function such that H is an MPC of F_1 over $[\underline{v}_1, v_a]$ and $H(v) \leq F_1(v_1) + \frac{(1-F_1(v_1))(v-v_a)}{v_b-v_a}$ for $v \in [\underline{v}_2, v_a]$, and G_1^* is an MPC of F_1 over $[v_a, \overline{v}_1^*]$.

Figure 10 depicts the equilibrium in Proposition 11, with $H(v) = F(v)$. Since the interval $v < v_a$ is not in $supp(G_2^*)$, H does not need to be linear for $v < v_a$, and can even be discontinuous. the only condition on H is that, for $v \in [\underline{v}_2, v_a]$, it must be below the line from G_1^* for $v \in [v_a, v_b]$. This condition ensures that firm 2 has no incentive to put any mass on $v < v_a$.

Proof of Proposition 11. Recall that $\overline{v}_i^* = \sup supp(G_i^*)$ and $\underline{v}_i^* = \inf supp(G_i^*)$. We characterize the equilibrium using the following steps:

Step 1. *(a) $\overline{v}_1^* = \overline{v}_2^* \equiv \overline{v}^*$ and (b) $\overline{v}^* < \overline{v}_1$:* To show part (a), suppose to the contrary that $\overline{v}_i^* < \overline{v}_j^*$ for some $i, j = 1, 2$. Then it is obviously profitable for firm j to move the probability mass around $\overline{v}_i^*$ and assigns it to $\overline{v}_i^*$ in the way of mean-preserving contraction. Part (b) follows from Lemma 15: Since F_1 is concave, there cannot exist G_1^* which is convex over $[\max\{\underline{v}_1^*, \underline{v}_2^*\}, \overline{v}_1]$ and is an MPC of F_1.

Step 2. *$\underline{v}_1^* \leq \underline{v}_2^*$:* Suppose to the contrary that $\underline{v}_1^* > \underline{v}_2^*$. Since $\underline{v}_1 \geq \underline{v}_2$, it follows that $\underline{v}_1^* > \underline{v}_1$, and $G_1^* \neq F_1$ over $[\underline{v}_1, \underline{v}_1^*]$. Let $\hat{v} = \inf\{v : G_1^*$ is an MPC of F_1 over $[\underline{v}_1, v]\}$. Then since F_1 is

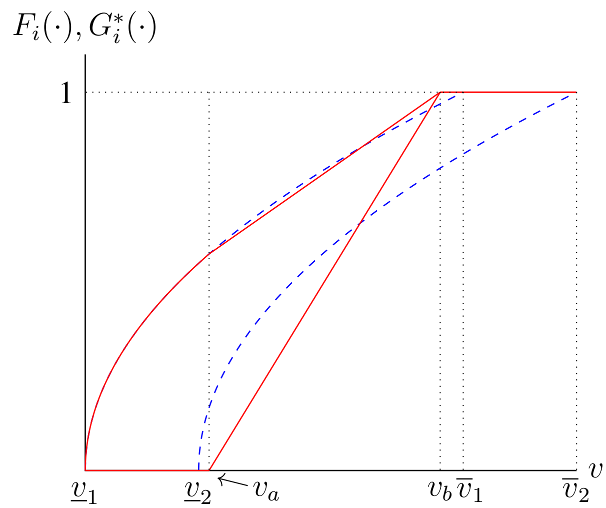


Figure 10: An equilibrium in the advertising game when both F_1 and F_2 are concave. The distribution functions in this figure are $F_1(v) = \sqrt{v}$ and $F_2(v) = F_1(v - 0.3)$.

concave, Lemma 15 implies that $\hat{v} = \overline{v}_1$. Then by Lemma 16, both G_1^* and G_2^* are linear over $[\underline{v}_1^*, \overline{v}^*]$. However, it follows that $\mathbb{E}_{G_1^*}[v] > \mathbb{E}_{G_2^*}[v]$, which contradicts the assumption that F_2 first-order stochastically dominates F_1.

Let $v_i^{(k)} = \sup\{v : G_i^* \text{ is an MPC of } F_2 \text{ over } [v, v_i^{(k-1)}]\}$, where $v_i^{(0)} = \overline{v}_i$.

Step 3. *$v_2^{(1)} = \underline{v}_2$:* Suppose to the contrary that $v_2^{(1)} > \underline{v}_2$. Then it must be that $G_2^*(v_2^{(1)}) = F_2(v_2^{(1)})$ and that $G_2^*(v_2^{(1)} + \varepsilon) < F_2(v_2^{(1)} + \varepsilon)$ for small $\varepsilon > 0$. However, since $F_2(v)$ is concave, and since Step 1 and Lemma 15 imply that $G_2^*(v)$ must be convex over its support, G_2^* cannot be an MPC of F_2, leading to a contradiction.

Step 4. *$v_1^{(1)} = \underline{v}_2^* = v_a$:* By Lemma 16, G_1^* is an MPC of F_1 over $[\underline{v}_2^*, \overline{v}^*]$. Now suppose to the contrary that $v_1^{(1)} > \underline{v}_2^*$, which implies that $G_1^*(v_1^{(1)}) = F_1(v_1^{(1)})$ and that G_1^* is an MPC of F_1 over $[\underline{v}_2^*, v_1^{(1)}]$. However, Lemma 16 implies that G_1^* must be linear over $[\underline{v}_2^*, v_1^{(1)}]$, and since F_1 is concave, G_1^* cannot be an MPC of F_1 over $[\underline{v}_2^*, v_1^{(1)}]$, a contradiction.

Step 1-4 determines the equilibrium advertising strategy G_2^* for all v and G_1^* for $v \geq v_a$. Finally, Step 5 provides a sufficient condition of G_1^* for $v < v_a$.

Step 5. *For $v \in [\underline{v}_2, v_a)$, $G_1^*(v) \leq F_1(v_1) + \frac{(1-F_1(v_1))(v-v_a)}{v_b - v_a}$:* Suppose to the contrary that there exists $\hat{v} \in [\underline{v}_2, v_a)$ such that $G_1^*(\hat{v}) > F_1(v_1) + \frac{(1-F_1(v_1))(\hat{v}-v_a)}{v_b - v_a}$. Then it is straightforward to show that it is a profitable deviation for firm 2 to take a probability mass on some $v \in [v_a, v_b)$ and put it on $\hat{v}$ and $v' \in (v, v_b)$ in the way of mean-preserving spread. ■

I.2 Convex F_i

Now consider a case in which both F_1 and F_2 are convex. First, note that if $\overline{v}_1 = \overline{v}_2$, Proposition 8 implies that there exists a full information equilibrium. Also, it is straightforward to show that the full information equilibrium is essentially the unique equilibrium of the advertising game: In any equilibrium, $G_i^*(v) = F_i(v)$ for $i = 1, 2$ and $v \geq \underline{v}_2$.

For the case of $\overline{v}_1 < \overline{v}_2$, the equilibrium structure is not trivial. The next proposition states our finding.

Proposition 12 *Suppose that F_1 and F_2 are both convex and that $\overline{v}_1 < \overline{v}_2$. Then for some $K \geq 2$, there exist sequences $\left(\{v_i^{(k)}\}_{k=0}^K\right)_{i=1,2}$ ($v_i^{(0)} = \overline{v}_i$, $v_i^{(K)} = \underline{v}_i$ for $i = 1, 2$, and $v_2^{(k+1)} < v_1^{(k)} < v_2^{(k)}$ for $k = 0, \ldots, K-1$), $\{\alpha_k\}_{k=1}^K$ and $\{\beta_k\}_{k=1}^{K-1}$ ($0 < \alpha_K < \cdots < \alpha_1, 0 < \beta_{K-1} < \cdots < \beta_1$) such that any equilibrium of the advertising game has the following structure:*

$$G_1^*(v) = \begin{cases} H(v) & \text{if } v \in [\underline{v}_1, v_1^{(K-1)}), \\ F_1(v_1^{(K-1)}) + \alpha_n(v - v_1^{(K-1)}) & \text{if } v \in [v_1^{(K-1)}, v_2^{(K-1)}), \\ F_1(v_1^{(K-1)}) + \alpha_n(v_2^{(K-1)} - v_1^{(K-1)}) & \\ \quad + \sum_{l=k+1}^{K-1} \alpha_l(v_2^{(l-1)} - v_2^{(l)}) + \alpha_k(v - v_2^{(k)}) & \text{if } v \in [v_2^{(k)}, v_2^{(k-1)}), k = 2, \ldots, n-2 \\ F_1(v_1^{(K-1)}) + \alpha_K(v_2^{(K-1)} - v_1^{(K-1)}) & \\ \quad + \sum_{l=2}^{K-1} \alpha_l(v_2^{(l-1)} - v_2^{(l)}) + \alpha_1(v - v_2^{(1)}) & \text{if } v \in [v_2^{(1)}, \overline{v}_1], \end{cases}$$

$$G_2^*(v) = \begin{cases} 0 & \text{if } v < v_1^{(K-1)}, \\ \sum_{l=k+1}^{K-1} \beta_l(v_1^{(l-1)} - v_1^{(l)}) + \beta_k(v - v_1^{(k)}) & \text{if } v \in [v_1^{(k)}, v_1^{(k-1)}), k = 1, \ldots, K-1 \\ 1 & \text{if } v \geq \overline{v}_1, \end{cases}$$

where

- *G_i^* is an MPC of F_i over $[v_i^{(k+1)}, v_i^{(k)}]$ for $k = 0, \ldots, K-1$,*
- *H is an increasing and right-continuous function such that $H(v) \leq F_1(v_1) + \alpha_K(v - v_1^{(K-1)})$ for $v \in [\underline{v}_2, v_1^{(K-1)})$,*
- *$G_1^*(\overline{v}_1) = F_1(v_1) + \alpha_K(v_2^{(K-1)} - v_1^{(K-1)}) + \sum_{l=2}^{K-1} \alpha_l(v_2^{(l-1)} - v_2^{(l)}) + \alpha_1(\overline{v}_1 - v_2^{(1)}) = 1.$*

Roughly speaking, the equilibrium in Proposition 12 has an "alternating piecewise linear" structure. Not only that both G_i^* are piecewise linear over $\cap_i supp(G_i^*) = [v_1^{(K-1)}, \overline{v}_1]$, the thresholds of each linear part ($v_i^{(k)}$) are alternating ($v_2^{(k+1)} < v_1^{(k)} < v_2^{(k)}$ for $k = 0, \ldots, K-1$). Moreover, G_i^* is an MPC of F_i over an interval where G_j^* is linear. Figure 11 depicts the equilibrium with

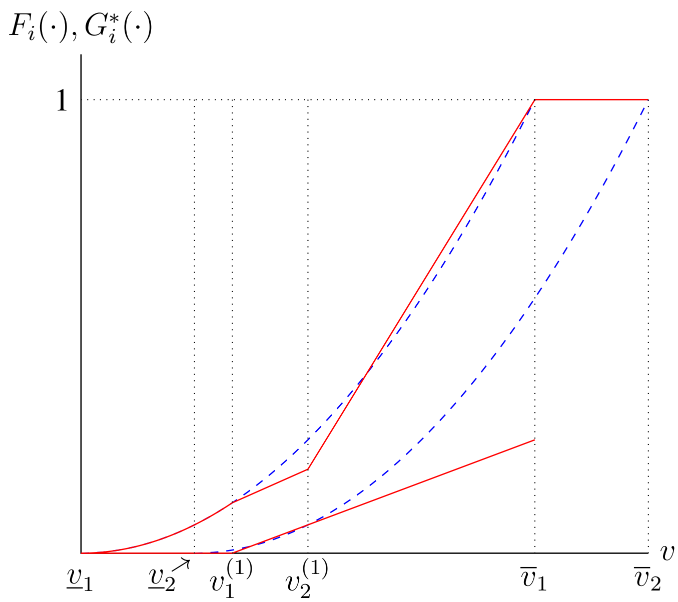


Figure 11: Equilibrium in the advertising game when both F_1 and F_2 are convex. The distribution functions in this figure are $F_1(v) = v^2$ and $F_2(v) = F_1(v - 0.25)$.

$K = 2$. Note that G_2^* has a probability mass at $\overline{v}_1$; we will show in the proof below that this property is a necessary condition for the equilibrium strategy.

Proof of Proposition 12. We characterize the equilibrium using the following steps:

Step 1. *(a)* $\overline{v}_1^* = \overline{v}_2^* \equiv \overline{v}^*$*, and (b)* $\overline{v}^* = \overline{v}_1$*:* The proof of part (a) is identical to that of the concave case. To show part (b), suppose to the contrary that $\overline{v}^* < \overline{v}_1$. Then, there must be no atom of probability at $v = \overline{v}^*$ for both G_1^* and G_2^*. Since $\overline{v}^* < \overline{v}_i$ for all i, it follows that $v_1^{(1)} < \overline{v}^*$ and $v_2^{(1)} < \overline{v}^*$. Assume $v_i^{(1)} \leq v_j^{(1)}$, then Lemma 16 implies that G_j^* must be linear for $[v_i^{(1)}, \overline{v}^*]$. However, since F_j is convex, we cannot construct G_j^* which is an MPC of F_j over $[v_j^{(1)}, \overline{v}^*]$, leading to a contradiction.

Step 2. $v_2^{(1)} < v_1^{(0)} < v_2^{(0)}$ *and* $v_2^{(1)} > \underline{v}_2$*:* It is straightforward that $v_2^{(1)} < v_1^{(0)} < v_2^{(0)}$. To prove the second part, suppose to the contrary that $v_2^{(1)} = \underline{v}_2$. Recall that $\underline{v}_2^* = \inf \mathit{Supp}(G_2^*) \geq \underline{v}_2$. Then since G_2^* is not linear on the neighborhood around $\underline{v}_2^*$, Lemma 16 implies that G_1^* must be an MPC of F_1 over $[\underline{v}_2^*, \overline{v}_1]$. However, since G_2^* must be linear over $[\underline{v}_2, \overline{v}_1]$ and since F_2 is convex, G_2^* cannot be MPC of F_1 over $[\underline{v}_2^*, \overline{v}_1]$, leading to a contradiction.

Step 3. *Suppose that* $\underline{v}_2 < v_2^{(l)} < v_1^{(l-1)} < v_2^{(l-1)}$*, then* $v_2^{(l+1)} < v_1^{(l)} < v_2^{(l)}$*. Furthermore, if* $v_2^{(l+1)} = \underline{v}_2$*, then* $v_1^{(l)} = \underline{v}_2^*$*:*

(a) $v_1^{(l)} < v_2^{(l)}$ *and* $v_1^{(l)} > \underline{v}_2$: Suppose to the contrary that $v_1^{(l)} \geq v_2^{(l)}$. Then since $v_1^{(l-1)} < v_2^{(l-1)}$, Lemma 16 implies that G_1^* must be linear over $[v_1^{(l)}, v_1^{(l-1)}]$. However, since F_1 is convex, G_1^*

cannot be an MPC of F_1 over $[v_1^{(1)}, v_1^{(l-1)}]$, a contradiction. To show the second part, suppose to the contrary that $v_1^{(l)} \leq \underline{v}_2$. Then similar to the first part, Lemma 16 implies that G_2^* must be linear and an MPC of F_2 over $[\underline{v}_2, v_2^{(l)}]$, which is not possible given the convexity of F_2.

(b) $v_2^{(l+1)} < v_1^{(l)}$: Suppose to the contrary that $v_2^{(l+1)} \geq v_1^{(l)}$. Note that by part (a), $v_2^{(l+1)} > \underline{v}_2$. Then since $v_2^{(l)} < v_1^{(l-1)}$, Lemma 16 implies that G_2^* must be linear over $[v_2^{(l)}, v_2^{(l-1)}]$. However, since F_2 is convex, G_2^* cannot be an MPC of F_2 over $[v_2^{(l)}, v_2^{(l-1)}]$, a contradiction.

(c) *If $v_2^{(l+1)} = \underline{v}_2$, then $v_1^{(l)} = \underline{v}_2^*$*: Suppose to the contrary that $v_1^{(l)} \neq \underline{v}_2^*$. It cannot be that $v_1^{(l)} < \underline{v}_2^*$, since G_2^* is not linear around $\underline{v}_2^*$, and thus not linear over $[v_1^{(l)}, v_1^{(l-1)}]$. Now suppose that $v_1^{(l)} > \underline{v}_2^*$. Then Lemma 16 implies that G_1^* must be linear and be an MPC of F_1 over $[\underline{v}_2^*, v_1^{(l)}]$, which is not possible given the convexity of F_1.

Step 4. *For $v \in [\underline{v}_2, v_1^{(K-1)})$, $G_1^*(v) \leq F_1(v_1) + \alpha_K(v - v_1^{(K-1)})$:* The proof of Step 4 is almost identical to that of Step 5 of the concave case. ■